Узлы и зацепления в трехмерном торе

Вьюнг Бао

Томский Государственный Университет, Томск

Научная конференция «Алгебраическая топология, гиперболическая геометрия и компьютерный анализ данных»

Томск, 08 Декабря 2023

A link L with n components in three-dimensional torus T^3 is an embedding of a disjoint union of n circle S^1 into three-dimensional torus. If $n = 1$ the link is called knot. Two link are considered equivalent if they are ambient isotopic, that is, if there exists a continuous deformation of T^3 which takes one link to the other.

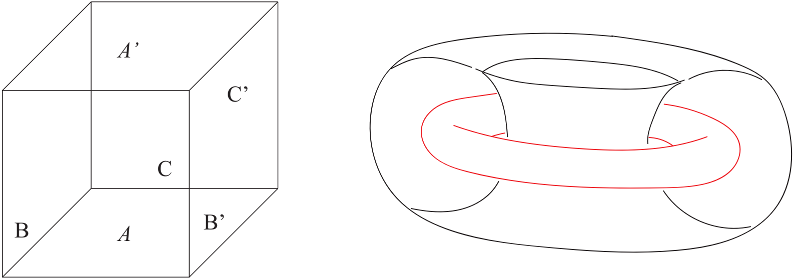


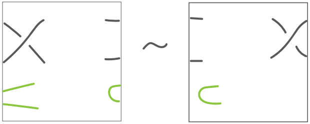
Рис.: 3-torus

A diagram of link in T^3 is a regular plane graph represented on a square (see Vuong), which has nodes of 4-valent (with extra structure representing the crossing in the link) and also 2-valent nodes (vertices with poles). One says that two such diagrams are equivalent if there is a sequence of generalised Reidemeister moves and vertex moves taking one diagram to the other. These moves are performed locally on the regular plane graph (with extra structure) that constitutes the link diagram.

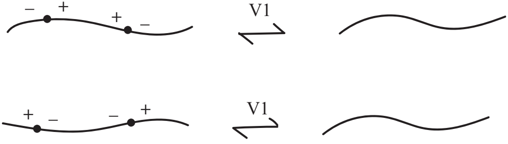
R1

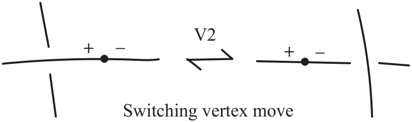
R2

R3

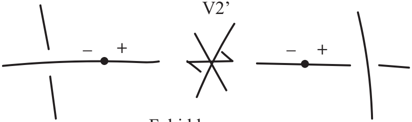


R4, R5 moving through boundary





Switching vertex move



Fobidden move

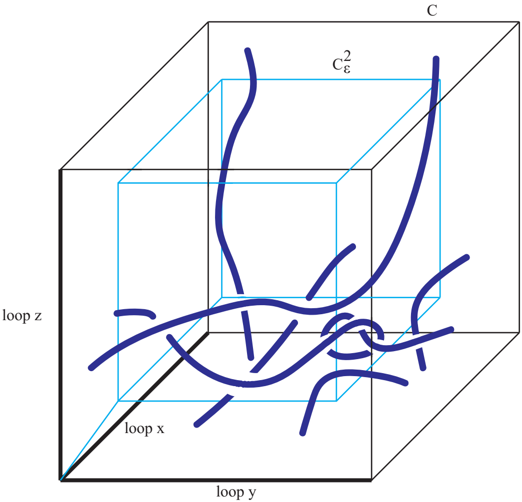


Рис.: A link in 3-torus

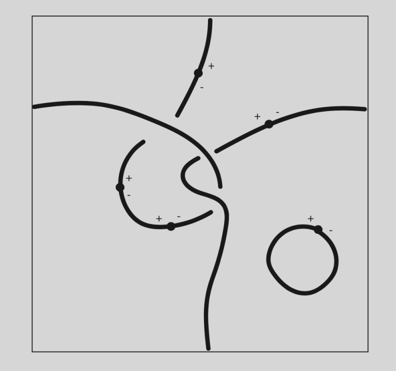


Рис.: Diagram

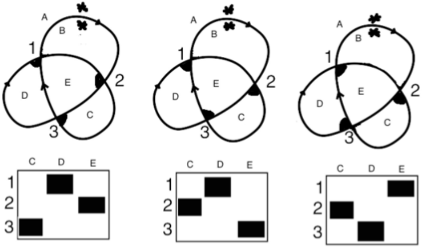
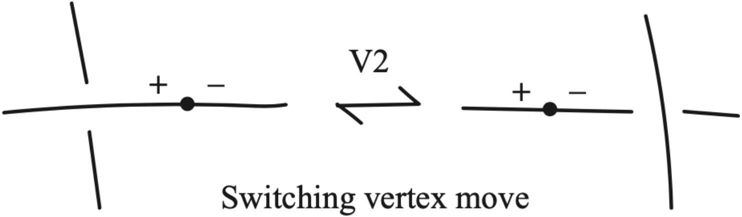
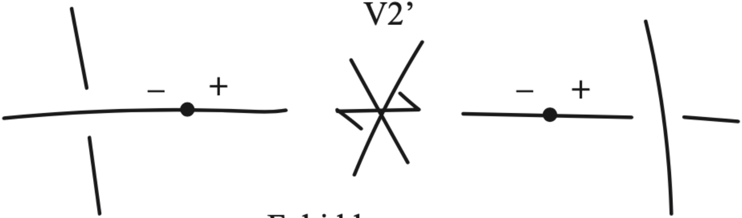


Рис.: State sum model

What am I looking for?
A state sum model for that type of formal diagrams of knots in 3-torus
The philosophy is any object and evaluation can be the Euler characteristic of some chain complexes.
And so there must be a homology theory behind the scene?!



Switching vertex move



Fobidden move

How can Boolean algebra be involved to this type of switching? Or the $\mathbb{Z}_2$ structure here could be applied?

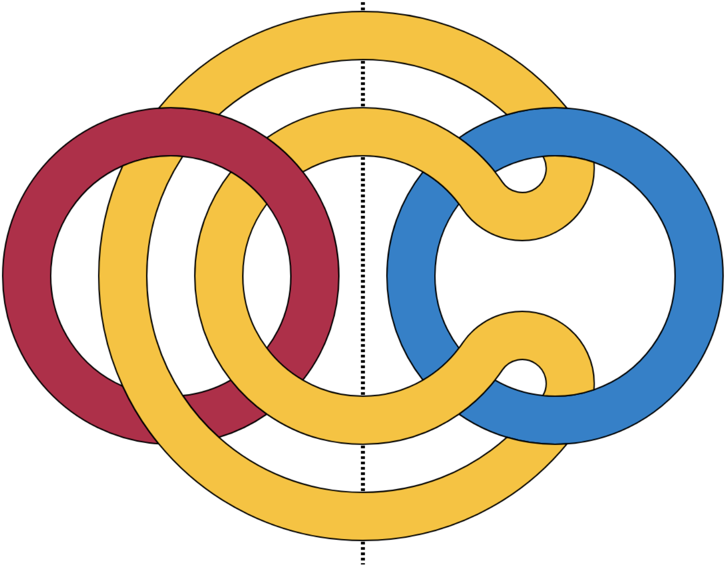
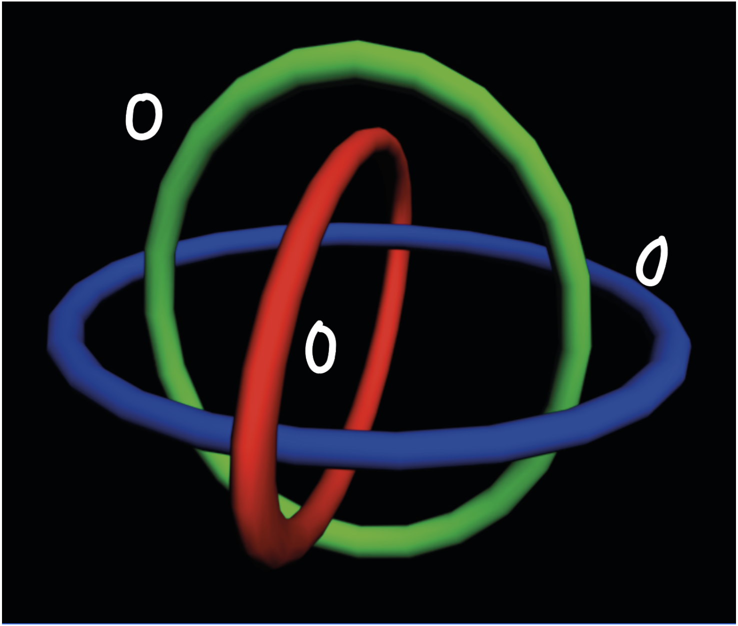


Рис.: Borromean rings


0
0
0

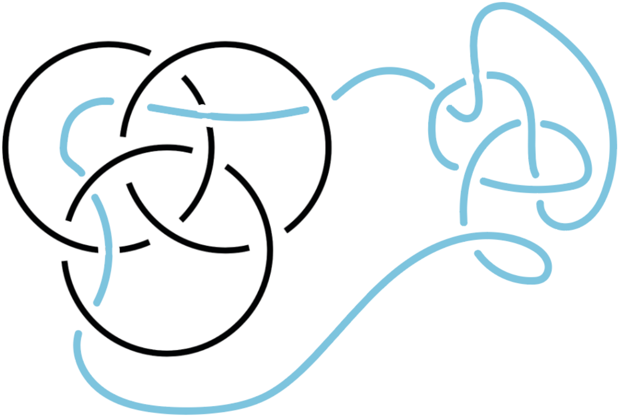


Рис.: Knots in 3-torus

I supposed an algorithm to get a presentation of the fundamental group of link complement from a diagram of the link as defined above. Having a diagram of a knot K in three dimensional torus we can easily define its homology class $[K] \in \mathbb{Z}^3$ of K. Also having a presentation of the fundamental group of link complement, the abelianization of the fundamental group $\pi_1(T^3 \backslash L)/[\pi_1(T^3 \backslash L), \pi_1(T^3 \backslash L)]$ is its first homology group $H_1(T^3 \backslash L)$.

By the diagrams with polarised vertices

W : $w_1, \ldots, w_s$ are the classical Wirtinger relations for each crossing, that is of the type $a_i a_j a_i^{-1} a_k^{-1} = 1$ or $a_i a_j^{-1} a_i^{-1} a_k^{-1} = 1$;

Q: Relations between loops corresponding to overpasses with identified endpoints on the boundary $x_i' = y\gamma_{k+1}^{-1} x_i \gamma_{k+1} y^{-1}$, $y_j' = x^{-1}\gamma_{k+1}^{-1} y_j \gamma_{k+1} x$, $\gamma_k^{-1} z_k' \gamma_k = z^{-1}\gamma_k^{-1} z_k \gamma_k z$

T: Torus relations $x_1^{\epsilon_1} \ldots x_n^{\epsilon_n} = zxz^{-1}x^{-1}$, $y_1^{\nu_1} \ldots y_m^{\nu_m} = yzy^{-1}z^{-1}$, $z_1^{\tau_1} \ldots z_l^{\tau_l} = yxy^{-1}x^{-1}$, when n, m or l is zero then the corresponding product define to be 1.

By the diagrams with surgery on Borromean rings
W : $w_1, ..., w_s$ are the classical Wirtinger relations for each crossing
S: Surgery relations $l_1 = 1; l_2 = 1; l_3 = 1$, corresponding to the three parallels of the Borromean rings in terms of generators as in classical diagram.

Рис.: Knots in 3-torus

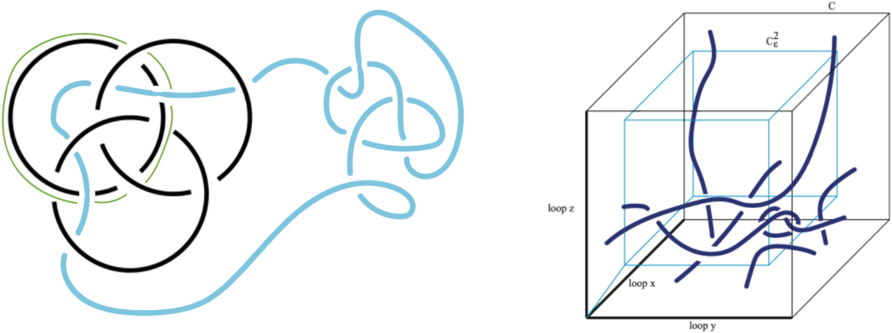


Рис.: Бурый

Theorem (V.)

Let L be a link in 3-torus T^3, with components $L_1, ..., L_\omega$. For each $\iota = 1, ..., \omega$, let $(\delta_\iota, \sigma_\iota, \xi_\iota) = [L_\iota] \in \mathbb{Z}^3 = H_1(T^3)$. Then

$$H_1(T^3 \backslash L) \cong \begin{cases} \mathbb{Z}^3 \oplus \mathbb{Z}_\rho, & \text{if } \omega = 1 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_\kappa \oplus \mathbb{Z}_\lambda, & \text{if } \omega = 2 \\ \mathbb{Z}^\omega \oplus \mathbb{Z}_\zeta \oplus \mathbb{Z}_\eta \oplus \mathbb{Z}_\theta, & \text{if } \omega \geq 3. \end{cases}$$

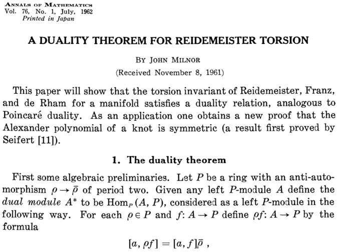
where $\rho = gcd(\delta_1, \sigma_1, \xi_1)$; κ and λ are the invariant factor of the matrix M_1; ζ, η and θ are the invariant factor of the matrix M_2.

$$M_1 = \begin{pmatrix} \delta_1 & \delta_2 \\ \sigma_1 & \sigma_2 \\ \xi_1 & \xi_2 \end{pmatrix};$$

$$M_2 = \begin{pmatrix} \delta_1 & \delta_2 & ... & \delta_\omega \\ \sigma_1 & \sigma_2 & ... & \sigma_\omega \\ \xi_1 & \xi_2 & ... & \xi_\omega \end{pmatrix}.$$

Now we see that the first homology group might contain torsion part. We say that a link $L \in T^3$ is nontorsion if torsion part $\mathrm{Tors}(H_1(T^3 \backslash L))$ is zero, otherwise we say that L is torsion. A local link or affine link is a link, which can be isotoped so that it's contained inside a 3-ball in T^3. A local link is clearly nontorsion.

The relation between Alexander polynomial of a knot and the torsion invariant of Reidemeister, Franz and de Rham for knot complement was first noticed by Milnor. As a consequence of the relation, Milnor gave another proof for symmetry of Alexander polynomial. Milnor applied the result to knot theory, considering the case of classical knot, i.e. the knot complement has the homology of the circle.



ANNALS OF MATHEMATICS
Vol. 76, No. 1, July, 1962
Printed in Japan

A DUALITY THEOREM FOR REIDEMEISTER TORSION

BY JOHN MILNOR

(Received November 8, 1961)

This paper will show that the torsion invariant of Reidemeister, Franz, and de Rham for a manifold satisfies a duality relation, analogous to Poincaré duality. As an application one obtains a new proof that the Alexander polynomial of a knot is symmetric (a result first proved by Seifert [11]).

1. The duality theorem

First some algebraic preliminaries. Let P be a ring with an anti-automorphism $\rho \to \bar{\rho}$ of period two. Given any left P-module A define the *dual module* A^* to be $\mathrm{Hom}_P(A, P)$, considered as a left P-module in the following way. For each $\rho \in P$ and $f: A \to P$ define $\rho f: A \to P$ by the formula

$$[a, \rho f] = [a, f]\bar{\rho} ,$$

It turns out that there are similar relations between Reidemeister torsion and twisted Alexander polynomial for the case of knot complement in other spaces, rather than three dimensional sphere when the homology group contains also torsion. The technology to get explicit relations as Milnor had created making use of simple homotopy theory for CW-complexes and Fox free differential calculus. Those ensure a CW structure for the knot complement, associated with a presentation of the fundamental group, so that the boundary maps are obtained by free derivatives. The method works out fine also for the case of knots and links in three dimensional torus.

Given a presentation of the group of a link, one may calculate its Alexander polynomial using Fox free calculus. We recall the following definition of Alexander polynomials. Let

$$P = \langle x_1, \dots, x_n | r_1, \dots, r_m \rangle$$

be a presentation of a group G and denote by $H = G/G'$ its abelianization. Let $F = \langle x_1, \dots, x_n \rangle$ be the corresponding free group. We apply the chain of maps

$$\mathbb{Z}F \xrightarrow{\frac{\partial}{\partial x}} \mathbb{Z}F \xrightarrow{\gamma} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}H,$$

where $\frac{\partial}{\partial x}$ denotes the Fox differential, γ is the quotient map by relations $r_1, \dots, r_m$ and α is the abelianization map. The Alexander-Fox matrix of the presentation P is the matrix $A = [a_{i,j}]$, where $a_{i,j} = \alpha(\gamma(\frac{\partial r_i}{\partial x_j}))$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. For $k = 1, \dots, \min\{m-1, n-1\}$, the k-th elementary ideal $E_k(P)$ is the ideal of $\mathbb{Z}H$, generated by the determinants of all the $(n-k)$ minors of A. The first elementary ideal $E_1(P)$ is the ideal of $\mathbb{Z}H$, generated by the determinants of the all the $(n-1)$ minors of A.

Definition

Let $L \subset S^3$ be a link, and let $E_k(P)$ be the k-th elementary ideal, obtained from a presentation P of fundamental group $\pi_1(S^3 \backslash L, *)$. Then the k-th link polynomial $\Delta_k(L)$ is the generator of the smallest principal ideal containing $E_k(P)$. The Alexander polynomial of L, denoted by $\Delta(L)$, is the first link polynomial of L.

For a classical link L in S^3, the abelianization of $\pi_1(S^3 \backslash L, *)$ is the free abelian group, whose genereators correspond to the components of L. For a link in 3-torus T^3, the abelianization of its link group may also contain torsion. In this case, the Alexander polynomial is not defined, we need the notion of twisted Alexander polynomials. Thus we recall the definition of twisted Alexander polynomials.

Let G be a group with a finite presentation P and ablianizatioon $H = G/G'$ and denote $K = H/Tors(H)$. Then every representation $\phi : Tors(H) \to \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ determines a twisted Alexander polynomial $\Delta^{\phi}(P)$ as follows. Choosing a splitting $H = Tors(H) \times K$, ϕ induces a ring homomorphism $\phi : \mathbb{Z}[H] \to \mathbb{C}[K]$ sending $(f, g) \in Tors(H) \times K$ to $\phi(f)g$. The ring homomorphism is called twisted homomorphism. Thus we apply the chain of maps

$$\mathbb{Z}[F] \xrightarrow{\frac{\partial}{\partial x}} \mathbb{Z}[F] \xrightarrow{\gamma} \mathbb{Z}[G] \xrightarrow{\alpha} \mathbb{Z}[H] \xrightarrow{\phi} \mathbb{C}[K]$$

and obtain the ϕ-twisted Alexander matrix $A^{\phi} = \left[\phi(\alpha(\gamma(\frac{\partial r_i}{\partial x_j})))\right]$. The twisted Alexander polynomial is then defined by $\Delta^{\phi}(P) = \gcd(\phi(E_1(P)))$.

Definition

Let $L \subset T^3$ be a link in the three dimensional torus T^3. For any presentation P of the link group $\pi_1(T^3 \backslash L, *)$, we may define the following. The Alexander polynomial of L, denoted by $\Delta(L)$, is the generator of the smallest principal ideal containing $E_1(P)$.
For any homomorphism $\phi : Tors(H_1(T^3 \backslash L)) \to \mathbb{C}^*$, the ϕ-twisted Alexander polynomial of L is $\Delta^{\phi}(L) = gcd(\phi(E_1(P)))$.

We know that the torsion subgroup of $H_1(T^3 \backslash L))$ is the group $\mathbb{Z}_\zeta \oplus \mathbb{Z}_\eta \oplus \mathbb{Z}_\theta$ in general. The ϕ-twisted Alexander polynomial $\Delta^{\phi}(L) \in \mathbb{Z}[\Omega][K]$ is defined up to multiplication by a unit.

Reidemeister torsion of cell complex

Let $\mathbb{F}$ be a field, V be a k-dimensional vector space over $\mathbb{F}$. Suppose that $b = (b_1, b_2, ..., b_k)$ and $c = (c_1, c_2, ..., c_k)$ are two bases of V then there is a non-singular $k \times k$ matrix (a_{ij}) such that $b_j = \sum_{i=1}^{k} a_{ij} c_i$. We write $[b/c] = \det(a_{ij}) \in \mathbb{F}^*$. Two bases b and c are said to have the same orientation if $[b/c] > 0$, and to be equivalent if $[b/c] = 1$.

Let $0 \to C \xrightarrow{\alpha} D \xrightarrow{\beta} E \to 0$ be a short exact sequence of vector spaces. Let $c = (c_1, c_2, ..., c_k)$ be a basis for C and $e = (e_1, e_2, ..., e_l)$ be a basis for E. Since the map β is surjective we can lift e_i to a vector $\tilde{e}_i$ in D. Then $ce = (c_1, ..., c_k, \tilde{e}_1, ..., \tilde{e}_l)$ is a basis for D and its equivalent class depends not on the choice of $\tilde{e}_i$ but only on the equivalence classes of c and e.

The finite chain complex

$(C, \partial) = (0 \to C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0)$ of finite-dimensional vector spaces over $\mathbb{F}$ is called acyclic if it is exact. The chain is called based if for each C_i a basis is chosen.

Assume that (C, ∂) is acyclic and based with basis c. Choose a basis b_i for $B_i = \operatorname{Im} \partial_{i+1} = \ker \partial_i$. From the short exact sequence $0 \to B_i \to C_i \to B_{i-1} \to 0$ we get a basis $b_i b_{i-1}$ for C_i.

Definition

The torsion of the acyclic and based chain complex C is defined to be $\tau(C) = \prod_{i=0}^{m} [b_i b_{i-1}/c_i]^{(-1)^{i+1}} \in \mathbb{F}$. If C is not acyclic then $\tau(C)$ is defined to be 0.

The torsion $\tau(C)$ depends on c but does not depend on the choice of b_i's. If a basis c_i' is used instead of c_i then the torsion is multiplied with $[c_i/c_i']^{(-1)^{i+1}}$.
Let X be a finite connected CW-complex and let $\pi = \pi_1(X)$. The universal cover $\tilde{X}$ of X has canonical CW-complex structure obtained by lifting the cells of X. If $\{e_i^k, 1 \leq i \leq n_k\}$ is an ordered set of oriented k-cells of X and $\tilde{e}_i^k$ is any lift of e_i^k then the ordered set $\{\tilde{e}_i^k, 1 \leq i \leq n_k\}$ is a basis of the $\mathbb{Z}[\pi]$-module $C_i(\tilde{X})$.

If $\mathbb{Z}[\pi] \xrightarrow{\phi} \mathbb{F}$ is a ring homomorphism then by the change of rings construction $\mathbb{F} \otimes C_*(\tilde{X})$ is a chain complex of finite dimensional vector spaces over $\mathbb{F}$. If this chain complex is acyclic then its torsion $\tau(\mathbb{F} \otimes C_*(\tilde{X})) \in \mathbb{F}*$ is defined. However $\tau(\mathbb{F} \otimes C_*(\tilde{X}))$ depends on the chosen of basis for $C_*(\tilde{X})$, that is on the choices of lifting cells $\{\tilde{e}_i^k, 1 \leq i \leq n_k\}$. If we fix a choice of a set of lifting cells as a basis for the $\mathbb{Z}[\pi]$-module $C_i(\tilde{X})$ but change the order of the cells in the basis then $\tau(\mathbb{F} \otimes C_*(\tilde{X}))$ is multiplied with ± 1. If we change the orientations of the cells then torsion is also multiplied with ± 1. If we choose a different lifting cell for e_i^k – by an action $h.\tilde{e}_i^k$ of a covering transformation $h \in \pi$ – then torsion is multiplied with $\phi(h)^{\pm 1}$.

Definition

The Reidemeister torsion $\tau^{\phi}(X)$ of the CW-complex X is defined to be the image of $\tau(\mathbb{F} \otimes C_*(\tilde{X}))$ under the quotient map $\mathbb{F} \to \mathbb{F}/ \pm \phi(\pi)$.

It is well known that torsion is a simple homotopy invariant and a topological invariant of compact connected CW-complexes. And for every topological manifold of dimension 3 admits a piecewise linear structure or in other words admits a triangulation. Such a piecewise linear structure is unique in the sense that every homeomorphism h between two piecewise linear manifolds is isotopic to a piecewise linear homeomorphism. In terms of triangulations, the triangulations can be subdivided so that there is an isomorphism of the subdivided triangulations isotopic to h. Thus torsion is well-defined for our cases.

Remark. We see, that for defining the twisted Alexander polynomial we need a representation $\phi : \mathit{Tors}(H) \to \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ of the torsion part $\mathit{Tors}(H)$ into $\mathbb{C}^*$ as described in the section 3. The representation induces the twisted homomorphism $\mathbb{Z}[H] \xrightarrow{\phi} \mathbb{C}[K]$, that we also denote by ϕ. If $\mathbb{Q}(K)$ denotes the field of quotient of $\mathbb{C}[K]$. Then by composing with the projection into the quotient, twisted homomorphism ϕ determines a ring homomorphism from $Z[H]$ to the field $\mathbb{Q}(K)$ that we still denote by ϕ. Thus with a representation $\phi : \mathit{Tors}(H) \to \mathbb{C}^*$ we can define both a twisted Alexander polynomial Δ^ϕ and a torsion τ^ϕ.

Reidemeister torsion of link complements in 3-torus

Let L be a link in three dimensional torus T^3. The Euler characteristic of 3-torus T^3 is $\chi(T^3) = 0$. Removing a tubular neighborhood $N(L)$ of the link L from 3-torus T^3 we obtain a compact 3-manifold X with boundary, that is the link complement in 3-torus. In terms of Euler characteristic, we have $0 = \chi(T^3) = \chi(X \cup N(L)) = \chi(X) + \chi(N(L)) - \chi(X \cap N(L))$, that implies $\chi(X) = 0$.

The complement X, then by pushing in one free face at a time, we can collapse X down to a 2-dimensional subcomplex Y, so X is simple homotopic to Y (see Whitehead). The 2-cell complex Y is of Euler characteristic zero. We can ensure that Y has a cellular structure, containing only one 0-cell σ^0; n 1-cells $\sigma^1_1, \ldots, \sigma^1_n$, m 2-cells $\sigma^2_1, \ldots, \sigma^2_m$, where $m = n - 1$.

The boundary maps are $\partial_1 = 0$ and $\partial_2(\sigma^2_i) = r_i$, where r_i is a word in σ^1_j, giving a presentation of fundamental group as $\pi = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$.

A derivation from a paper by Huynh and Le

Let $\tilde{Y}$ be the maximal abelian cover of Y. The cellular complexes of $\tilde{Y}$ is considered as modules over integral group ring $\mathbb{Z}(H)$, where H is the first homology group of Y. We have a chain complex of $\mathbb{Z}(H)$-modules

$$C_2(\tilde{Y}) \xrightarrow{\partial_2} C_1(\tilde{Y}) \xrightarrow{\partial_1} C_0(\tilde{Y}) \to 0$$

The boundary maps are obtained by Fox's free differential calculus (Compare Fox p. 547, Milnor p. 146): $\partial_1(\tilde{\sigma}_i^1) = pr(x_i - 1)\tilde{\sigma}^0$ and $\partial_2(\tilde{\sigma}_i^2) = \sum_{j=1}^n pr(\frac{\partial r_i}{\partial x_j})\tilde{\sigma}_j^1$, where the tilde sign denotes a lift of the cell to $\tilde{Y}$. The natural projection pr is the composition of the maps α, γ in the chain $\mathbb{Z}[F] \xrightarrow{\gamma} \mathbb{Z}[G] \xrightarrow{\alpha} \mathbb{Z}[H]$.

Fix a splitting of H as a product $H = K \times Tors(H)$ of the free part $K = H/Tors(H)$ and the torsion part $Tors(H)$. Denote the quotient field $Q(\mathbb{C}[K])$ of $\mathbb{C}[K]$ by $\mathbb{Q}(K)$. Using the homomorphism $\phi : \mathbb{Z}[H] \to \mathbb{C}[K] \hookrightarrow \mathbb{Q}(K)$, construct the tensor $\mathbb{Q}(K) \otimes_{\mathbb{Z}[H]} C_i(\tilde{Y})$, considered as a vector space over $\mathbb{Q}(K)$. We have a chain complex of vector spaces over $\mathbb{Q}(K)$:

$$C = (\mathbb{Q}(K) \otimes_{\mathbb{Z}[H],\phi} C_2(\tilde{Y}) \xrightarrow{\partial_2} \mathbb{Q}(K) \otimes_{\mathbb{Z}[H],\phi} C_1(\tilde{Y})$$

$$\xrightarrow{\partial_1} \mathbb{Q}(K) \otimes_{\mathbb{Z}[H],\phi} C_0(\tilde{Y}) \to 0).$$

The boundary maps are $[\partial_1]_i = \phi(x_i) - 1$, and $[\partial_2]_{i,j} = \phi(\frac{\partial r_j}{\partial x_i})$, $1 \leq u < n$, $1 \leq j \leq n-1$. Let $A = [\partial_2]^t$.
Denote the columns of A by u_i, $1 \leq i \leq n$, and denote the $(n-1) \times (n-1)$ matrix obtained from A by omitting the column u_i by A_i.
Since C is a chain we have $0 = \partial_1(\partial_2(\tilde{\sigma}_i^2)) = (\sum_{j=1}^n \phi(\frac{\partial r_i}{\partial x_j})(\phi(x_j) - 1))\tilde{\sigma}^0$, thus $\sum_{j=1}^n \phi(\frac{\partial r_i}{\partial x_j})(\phi(x_j) - 1) = 0$. That means $\sum_{j=1}^n (\phi(x_j) - 1)u_j = 0$. For any $i > j$ we have

$$\begin{aligned}
(\phi(x_j) - 1)\det A_i &= \det[u_1, ..., u_{j-1}, (\phi(x_j) - 1)u_j, u_{j+1}, ..., \hat{u}_i, ..., u_n] \\
&= \det[u_1, ..., u_{j-1}, -\sum_{k \neq j}(\phi(x_k) - 1)u_k, u_{j+1}, ..., \hat{u}_i, ..., u_n] \\
&= (-1)^{i-j+1}(\phi(x_i) - 1)\det A_j.
\end{aligned}$$

Thus for any i and j,

$$(\phi(x_i) - 1)\det A_j = \pm(\phi(x_j) - 1)\det A_i. \tag{1}$$

Because H has at least three free generators (Theorem 1), the image $\phi(\pi)$ cannot be $\{1\}$, thus there is at least one x_i such that $\phi(x_i) \neq 1$. The property $\partial_1(\tilde{\sigma}_i^1) = (\phi(x_i) - 1)\tilde{\sigma}^0$ implies $\partial_1(\frac{1}{\phi(x_i)-1}\tilde{\sigma}_i^1) = \tilde{\sigma}^0$, so ∂_1 is onto. Therefore the chain C is exact if and only if ∂_2 is injective, which means the rank of its matrix is exactly $n-1$. Thus C is acyclic if and only if A has a nonzero $(n-1) \times (n-1)$ minor.

The Reidemeister torsion of C with respect to ϕ is the torsion $\tau^\phi(Y)$ of Y, and since torsion is a simple homotopy invariant, it is also the torsion $\tau^\phi(X)$ of X.

Now if we assume that C is acyclic. Take the standard bases of $\mathbb{Q}(K) \otimes_{\mathbb{Z}[H],\phi} C_*(\tilde{Y})$ given by $\tilde{\sigma}_j^i$ as above. A lift of $c_0 = \{\tilde{\sigma}_0\}$ is $\{\frac{1}{\phi(x_i)-1}\tilde{\sigma}_i^1\}$. Then

$$\tau^\phi(X) = [(\sum_{j=1}^{n} \phi(\frac{\partial r_1}{\partial x_j})\tilde{\sigma}_j^1, ..., \sum_{j=1}^{n} \phi(\frac{\partial r_{n-1}}{\partial x_j})\tilde{\sigma}_j^1, \frac{1}{\phi(x_i)-1}\tilde{\sigma}_i^1)/(\tilde{\sigma}_1^1, ..., \tilde{\sigma}_n^1)]$$

$$= \frac{(-1)^{i+n}}{(\phi(x_i)-1)}\det A_j.$$

Thus if $\phi(x_i) \neq 1$ then $\tau^{\phi}(X) = \pm \mathrm{det} A_i/(\phi(x_i) - 1)$. By equation 1 if $\phi(x_j) = 1$ then $\mathrm{det} A_j = 0$, hence the following formula is correct for all i, whether C is acyclic or not:

$$(\phi(x_i) - 1)\tau^{\phi}(X) = \pm \mathrm{det} A_i \in \mathbb{Q}(K)/ \pm K. \tag{2}$$

Remark. The equation 2, derived in the work by Huynh and Le for links in projective space, is hold for links in lens space and for link complement in any space of Euler characteristic zero.

Theorem (V.)

The Reidemeister torsion and the twisted Alexander polynomial of the complement of a link in 3-torus are the same.

Proof

The cellular structure of the link complement X in 3-torus is simple homotopic to a 2-dimensional subcomplex Y of Euler characteristic zero, so the structure admits a presentation for the fundamental group of X with n generators and $m = n - 1$ relations. So the Alexander-Fox matrix A associated to such a presentation is a $(n-1) \times n$ matrix. So the twisted Alexander polynomial $\Delta^{\phi}(X)$ is defined to be the greatest common devisor $\gcd(\det A_1, ..., \det A_n)$ of all $(m-1)$-minor A_i of matrix A, obtained by removing the i-th column of A.
By equation (2), we have

$$\Delta^{\phi}(X) = \gcd(\det A_1, ..., \det A_n) =$$
$$= \gcd((\phi(x_1) - 1)\tau^{\phi}(X), ..., (\phi(x_n) - 1)\tau^{\phi}(X))$$

Case 1: L is a non-torsion knot or link. We have the first homology group of complement (see section 2) is $H_1(T^3 \backslash L) \cong \mathbb{Z}^3 \oplus \mathbb{Z}^\omega$, where ω is the number of components. Denote with $t_1, ..., t_{\omega+3}$ the generators of H_1.

Then $\phi(x_i) = t_1^{h_i^1} ... t_{\omega+3}^{h_i^{\omega+3}}$ for $i = 1, ..., n$. Let
$g = \mathrm{gcd}((\phi(x_1) - 1), ..., (\phi(x_n) - 1) \in \mathbb{Z}[t, t^{-1}]$.

For a moment we set $t_2 = ... = t_{\omega+3} = 1$. So g divides each of $(t_1^{h_i^1} - 1)$ for $i = 1, ..., \omega + 3$. Observe that (see Lickorish) for any $a, b \in \mathbb{Z}$

$$(t^a - 1) + t^a(t^b - 1) = t^{a+b} - 1$$

and

$$(t^a - 1) - t^{a-b}(t^b - 1) = t^{a-b} - 1.$$

Applying the argument we conclude that g divides $t_1^{\sum_{i=1}^n \alpha_i h_i^1} - 1$ for any $\alpha_i \in \mathbb{Z}$. Since t_1 is an element of canonical projection of fundamental group $pr(\pi)$ there is a collection of $\alpha_i \in \mathbb{Z}$ such that $t_1 = \prod_{i=1}^n pr(x_i^{\alpha_i}) = t_1^{\sum_{i=1}^n \alpha_i h_i^1}$. Thus g divides $(t_1 - 1)$. Now by letting $t_j = 1$ for $j \neq i, i = 2, .., (\omega + 3)$ and repeating the argument we obtain $g = \gcd((\phi(x_1) - 1), ..., (\phi(x_n) - 1) = \gcd((t_1 - 1), ..., (t_{\omega+3} - 1) = 1$

Case 2: L is a torsion link. The first homology group of link complement has free part of rank at most $\omega + 2$ and the torsion part might be a product of at most three cyclic group $\mathbb{Z}_\zeta \oplus \mathbb{Z}_\eta \oplus \mathbb{Z}_\theta$, where $\zeta, \eta, \theta \in \mathbb{N}$ are the order of respective groups. Without lost of generality we consider the case when the first homology group has the rank r and torsion part has the structure $\mathbb{Z}_\zeta \oplus \mathbb{Z}_\eta \oplus \mathbb{Z}_\theta$. Now denote with $t_1, ..., t_r$ the generators of free part F and u_1, u_2, u_3 are the generator of torsion part $Tors(H_1)$.

We have projection of x_i is $pr(x_i) = t_1^{h_i^1} \dots t_r^{h_i^r} u_1^{k_i^1} u_2^{k_i^2} u_3^{k_i^3}$. For some homomorphism $\phi : Tors(H_1) \to \mathbb{C}*$ the image $\phi(x_i)$ is defined to be $\phi(x_i) = t_1^{h_i^1} \dots t_r^{h_i^r} \phi(u_1^{k_i^1} u_2^{k_i^2} u_3^{k_i^3})$. Setting $t_2 = \dots = t_r = 1$, applying the previous reasoning we conclude that $g = \gcd((\phi(x_1) - 1), \dots, (\phi(x_n) - 1)$ divides $t_1^{\sum_{i=1}^n \alpha_i h_i^1} \phi(u_1^{\sum_{i=1}^n \alpha_i k_i^1} u_2^{\sum_{i=1}^n \alpha_i k_i^2} u_3^{\sum_{i=1}^n \alpha_i k_i^3}) - 1$ for any $\alpha_i \in \mathbb{Z}$. Since t_1 is an element of canonical projection of fundamental group $pr(\pi)$ there is a collection of $\alpha_i \in \mathbb{Z}$ such that $t_1 = \prod_{i=1}^n pr(x_i^{\alpha_i}) = t_1^{\sum_{i=1}^n \alpha_i h_i^1} \phi(u_1^{\sum_{i=1}^n \alpha_i k_i^1} u_2^{\sum_{i=1}^n \alpha_i k_i^2} u_3^{\sum_{i=1}^n \alpha_i k_i^3})$. So $\sum_{i=1}^n \alpha_i h_i^1 = 1$ and $\phi(u_1^{\sum_{i=1}^n \alpha_i k_i^1} u_2^{\sum_{i=1}^n \alpha_i k_i^2} u_3^{\sum_{i=1}^n \alpha_i k_i^3}) = 1$. Thus analogously we get $\gcd((\phi(x_1) - 1), \dots, (\phi(x_n) - 1) = 1$ that complete the proof.

[1] V. Q. Huynh, T. T. Q. Le, *Twisted Alexander polynomial of links in the projective space*, J. Knot Theory Ramifications, **17** (2008), 411–438.
[2] E. Horvat, Boštjan Gabrovšek, *The Alexander polynomial of links in lens spaces*, J. Knot Theory Ramifications, **28**:8 (2019), article number1950049.
[3] R. H. Fox, *A quick trip through knot theory*, Topology of 3-manifolds, MK, Fort Jr editor, Prentice-Hall, 1962.
[4] V.Turaev, *Torsion of 3-dimensional manifolds*, Birkhäuser Verlag, Basel-Boston-Berlin, 2002.
[5] J. H. C. Whitehead, *Simple homotopy types*, Amer. J. Math., **72** (1950), 1–57.
[6] B. Vuong, *Fundamental group and twisted Alexander polynomial of knots in 3-torus*, https://arxiv.org/abs/2302.10461
[7] W. B. R. Lickorish, *An introduction to knot theory* , Graduate Texts in Mathematics, vol. 175, Springer, 1997

Thank you!