On growth of multivalued dynamics

Zonov MN, Bardakov VG, Kozlovskaya TA

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n-valued groups

Consider the n-th symmetric power of a set: $\mathrm{Sym}^n(X) = X^n/S_n$.

An n-valued multiplication structure on X is a map

$$*\colon X \times X \to \mathrm{Sym}^n(X); \quad x * y = [z_1, \ldots, z_n] \in \mathrm{Sym}^n(X).$$

It naturally extends to a map

$$* : X \times \mathrm{Sym}^n(X) \to \mathrm{Sym}^{n^2}(X); \quad x * [z_1, \ldots, z_n] = x * z_1 \cup \ldots \cup x * z_n$$

n-valued groups

Definition

A set X equipped with an n-valued multiplication $*$ is called an *n-valued group* if it has the following properties

- Associativity: the n^2-sets $(x * y) * z$ and $x * (y * z)$ are equal for all $x, y, z \in X$.
- Unit: $e \in X$ such that for all $x \in X$ we have

$$e * x = x * e = [x, \ldots, x].$$

- Inverse: a map $\mathrm{inv} : X \to X$ such that

$$e \in x * \mathrm{inv}(x) \text{ and } e \in \mathrm{inv}(x) * x.$$

Examples

Example 1

Let G be a (1-valued) group. Consider $x * y = [xy, xy, \ldots, xy]$ and $\mathrm{inv}(g) = g^{-1}$.

Example 2

$X = \mathbb{N} \cup \{0\}$ with $e = 0$, $\mathrm{inv}(x) = x$ and the multiplication

$$x * y = \big[x + y, |x - y|\big].$$

Note that in order to verify associativity one has to check that

$$\big[x+y+z, |x-y-z|, x+|y-z|, |x-|y-z||\big] = \big[x+y+z, |x+y-z|, |x-y|+z, ||x-y|-z|\big].$$

Coset groups

Let G be a (1-valued) group and $A \subset \operatorname{Aut} G$ a finite group of automorphisms of order n.

Denote $X = G/A$, and denote by $\pi : G \to X$ the quotient map.

Now define the n-valued multiplication by the formula

$$\pi(x) * \pi(y) = \Big[\pi\big(xy^a\big)\Big| a \in A\Big].$$

Note that the multiplication does not depend on the specific choice of x and y.

Theorem

The set X with multiplication $*$, the unit $e = e_G$ and the inverse $\operatorname{inv}(\pi(x)) = \pi(x^{-1})$ is an n-valued group.

Coset groups

Example 3

Let $G = \mathbb{Z}$ and $A = \{\varepsilon, -\varepsilon\}$, where $x^{-\varepsilon} = -x$. Then G/A can be associated with non-negative integers, and the multiplication is given by the formula

$$|x| * |y| = [|x + y^{\varepsilon}|, |x + y^{-\varepsilon}|] = \Big[|x| + |y|, \big||x| - |y|\big|\Big],$$

which gives us the same 2-valued group as Example 2.

The same 2-valued group can also be obtained by taking

$$G = \langle a, b \mid a^2 = e, b^2 = e \rangle$$

with A swapping a and b.

Cyclic dynamics

Let X be an n-valued group and $z \in X$. Consider the action of z on symmetric powers of X by right multiplication:

$$T\colon \operatorname{Sym}^k(X) \to \operatorname{Sym}^{kn}(X); \quad T(y) = y * z.$$

Consider the map $\operatorname{Set}\colon \operatorname{Sym}(X) \to 2^X$ that maps an unordered tuple of elements of X into a subset of X consisting of all unique elements of that tuple. For $r \in \mathbb{N}$ and $y \in X$ denote by $\xi_y(r)$ the number of unique elements in $T^r(y)$:

$$\xi_y(r) = \left|\operatorname{Set}\left(T^r(y)\right)\right|.$$

The function $\xi_y\colon \mathbb{N} \to \mathbb{N}$ is called *the growth function* of the cyclic dynamic T in the point y.

Cyclic dynamics

Example 4

Consider the 2-valued group $\mathbb{Z}/\mathbb{Z}_2$ from Examples 2 and 3. For $z = 1$ let us compute the growth function ξ_0.

For the first few r the unordered tuple $T^r(0)$ looks as follows:

$$[1,1],\ [0,0,2,2],\ [1,1,1,1,3,3],\ [0,0,0,0,2,2,2,2,4,4]\dots$$

It can be proven that $\mathrm{Set}(T^r(0))$ consists of all even numbers form 0 to r if r is even, and all odd numbers from 0 to r if r is odd.

It follows that $\xi_0(r)$ is the integer part of $\frac{n+2}{2}$.

Growth in coset groups

Let M be an monoid with a generating set L. Any element $g \in M$ can be represented as a word over L. Denote by $l(g)$ the minimal length of such a word. For any non-negative r let $S(r)$ be the set of all elements g in M such that $l(g) = r$, and $B(r)$ the set of all elements g in M such that $l(g) \leqslant r$.

Theorem 1

Let $X = G/A$ be an n-valued coset group and $z \in X$. Then for the dynamic T, generated by z, the growth function ξ_y satisfies the following inequality:

$$\frac{1}{n}|S(r)| \leqslant \xi_y(r) \leqslant |B(r)|,$$

where $M = \langle z^a \colon a \in A \rangle \subset G$.

Growth in coset groups

Corollary

If G is a group with polynomial growth, then every element of the coset group G/A has polynomial growth.

Example

Let $G = F_2 \times \mathbb{Z}$ and A is a 2-element group acting on $\mathbb{Z}$ by inversion. Every element of the coset group G/A has polynomial growth.

A special case of 2-valued groups

Theorem 2

Let X be a 2-valued group such that $\mathrm{inv}(x) = x$ for every $x \in X$. Then for any $z \in X$ the growth function ξ_y satisfies the inequality

$$\xi_y(r) \leqslant r(r+1).$$

Cayley graphs of free groups

For an n-valued group X with a finite generating set S consider the graph $\Gamma(X, S)$, where the vertices are elements of X, and vertices x and y are connected by an edge if $x \in y * s$ or $y \in x * s$ for some $s \in S$.

Proposition 3

Let F be a nonabelian free group with a free generating set S and A a subgroup of permutations of S. Then the Cayley graph of the coset group F/A is quasi-isometric to the Cayley graph of F.

Thank you for your attention!