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CONFERENCE “ALGEBRAIC TOPOLOGY, HYPERBOLIC GEOMETRY AND COMPUTER DATA ANALYSIS” 5-9 DECEMBER 2023

MULTIVALUED GROUPS AND NEWTON POLYHEDRON

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National Research
Tomsk
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Supported by the federal program Priority 2030 and organised within the framework of the mirror labs

PUBLICATIONS

1. T.A. Kozlovskaya, Multi-groups, to appear.

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2. V.G. Bardakov, T.A. Kozlovskaya, Multivalued groups and Newton polyhedron. htttps://arxiv.org/abs/2305.07261.

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ПРИОРИТЕТ 2030

3. V.G. Bardakov, T.A. Kozlovskaya, D.V. Talalaev, Groups and quandles: n-valued and n-tuple, in progress.

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T.A. Kozlovskaya, Multi-groups, to appear.

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HOMOGENEOUS ALGEBRAIC SYSTEMS

Definition 1. Let $\mathcal{A} = (A, f_i, i \in I)$ be an algebraic system with a set of algebraic operations f_i of arity n_i. It is said to be m-homogeneous I-system if all arities n_i are equal to m.

In particular, if $|I| = n$, we will say on m-homogeneous n-system.

If $m = 2$ we will say on groupoid n-system.

HOMOGENEOUS ALGEBRAIC SYSTEMS

EXAMPLE 1. A ring $(K, +, \cdot)$ is groupoid 2-system.

EXAMPLE 2. Semigroup (monoid, group) system $\mathcal{G} = (G, *_i, i \in I)$, where $(G, *_i)$ is a semigroup (monoid, group) for any $i \in I$ are examples of semigroup (monoid, group) I-system.

2-HOMOGENEOUS 2-SYSTEMS

EXAMPLE 3. A semigroup system with two operations is a duplex. That is an algebraic system with two associative binary operations (without added connections between these operations).

Duplexes were introduced by T. Pirashvili.

T. Pirashvili *Sets with two associative operations*, Cent.Eur.J.Math.1(2), (2003), 169–183.

2-HOMOGENEOUS n-SYSTEMS

EXAMPLE 4. *n-Tuple semigroup* is an algebraic system

$$\mathcal{S} = (S, *_i, i \in I), \quad |I| = n,$$

such that $(S, *_i)$ is a semigroup for any $i \in I$ and with the following axiom which connects these operations,

$$(a *_i b) *_j c = a *_i (b *_j c), \quad a, b, c \in S, \quad i, j \in I.$$

n-Tuple semigroup were introduced by N. Koreshkov.

N. Koreshkov, *n-Tuple algebras of associative type*, Izv. Vyssh. Uchebn. Zaved. Mat., No. 12, 34–42 (2008).

2-HOMOGENEOUS 2-SYSTEMS

EXAMPLE 5. A dimonoid is a set X together with two binary operations $\vdash$ and $\dashv$ satisfying the following axioms:

$$\begin{cases} x \dashv (y \dashv z) \stackrel{1}{=} (x \dashv y) \dashv z \stackrel{2}{=} x \dashv (y \vdash z), \\ (x \vdash y) \dashv z \stackrel{3}{=} x \vdash (y \dashv z), \\ (x \dashv y) \vdash z \stackrel{4}{=} x \vdash (y \vdash z) \stackrel{5}{=} (x \vdash y) \vdash z. \end{cases}$$

for all x, y and $z \in X$. Observe that relations 1 and 5 are the "associativity" of the products $\vdash$ and $\dashv$ respectively.

Dimonoids were introduced by J.-L.Loday in his construction of a universal enveloping algebra for the Leibniz algebra.

J.-L.Loday, *Algebres ayant deux operations associatives (dialgebres)*. C. R. Acad. Sci. Paris Ser. I Math. 321 (1995), no. 2, 141–146.

DIMONOIDS

a) Let M be a monoid. Put $D = M \times M$ and define the products by

$$(m, n) \dashv (m', n') := (m, nm'n'),$$

$$(m, n) \vdash (m', n') := (mnm', n').$$

$D = (D, \dashv, \vdash)$ is a dimonoid.

b) Let G be a group and X be a G-set. The following formulas define a dimonoid structure on $X \times G$:

$$(x, g) \dashv (y, h) := (x, gh),$$

$$(x, g) \vdash (y, h) := (g \cdot x, gh).$$

We see that a dimonoid is an example of group system with 2 operations.

SKEW BRACES

EXAMPLE 6. A group system $(G, \cdot, \circ)$ with two operation is said to be a skew (left) brace if

$$g_1 \circ (g_2 \cdot g_3) = (g_1 \circ g_2) \cdot g_1^{-1} \cdot (g_1 \circ g_3)$$

for all $g_1, g_2, g_3 \in G$, where g_1^{-1} denotes the inverse of g_1 in $(G, \cdot)$.

We call $(G, \cdot)$ the additive group and $(G, \circ)$ the multiplicative group of the skew left brace $(G, \cdot, \circ)$. A skew left brace $(G, \cdot, \circ)$ is said to be a (left) brace if $(G, \cdot)$ is an abelian group.
Braces were introduced by W. Rump as an algebraic system related to the quantum Yang—Baxter equation. L. Guarnieri and L. Vendramin defined for the same purposes a more general notion of a skew left brace.

W. Rump, *Braces, radical rings and the quantum Yang–Baxter equations*, J. Algebra 307 (2007), 153–170.

L. Guarnieri L. and Vendramin, *Skew braces and the Yang–Baxter equation*, Math. Comp. 86 (2017), 2519–2534.

CONNECTION BETWEEN SKEW BRACES AND DIMONOIDS

Let $(G, \cdot)$ be a group.

PROPOSITION 1.

1) If $a \circ b = ab$ then $(G, \cdot, \circ)$ is a skew left brace . If $a \vdash b = a \dashv b = ab$ then we get a dimonoid.

2) If $a \circ b = ba$, then $(G, \cdot, \circ)$ is skew left brace. If $a \vdash b = ab$ and $a \dashv b = ba$ then $(G, \dashv, \vdash)$ is not a dimonoid.

T.A. Kozlovskaya, Multi-groups, to appear.

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CONNECTION BETWEEN SKEW BRACES AND DIMONOIDS

Let $(G, \cdot, \circ)$ be a brace and $a \vdash b = ab, a \dashv b = a \circ b$

QUESTION 4. Is $(G, \vdash, \dashv)$ a dimonoid?

The next example shows that in the general case the answer is negative.

EXAMPLE 1. Let us take the brace $(\mathbb{Z}, +, \circ)$, where $(\mathbb{Z}, +)$ is the infinite cyclic group and $a \circ b = a + (-1)^a b$, $a, b \in \mathbb{Z}$. Note that

$$a \circ b = a + (-1)^a b = \begin{cases} a + b, & \textit{if } a \textit{ is even}; \\ a - b, & \textit{if } a \textit{ is odd}. \end{cases}$$

Put

$$a \vdash b = a + b, a \dashv b = a \circ b.$$

CONNECTION BETWEEN SKEW BRACES AND DIMONOIDS

Let us check the next axiom of dimonoid:

$$a \dashv (b \dashv c) = a \dashv (b \vdash c).$$

If we take $a = 2$, $b = 3$, $c = 4$, then $a \dashv (b \dashv c) = a + b - c = 1$. On the other side $a \dashv (b \vdash c) = a + b + c = 9$.
Therefore, the skew brace $(\mathbb{Z}, +, \circ)$ is not a dimonoid.

CONNECTION BETWEEN SKEW BRACES AND DIMONOIDS

QUESTION 5. Under which conditions a skew brace $(G, \cdot, \circ)$ is a dimonoid with respect to the operations $a \vdash b = ab, a \dashv b = a \circ b$?

QUESTION 6. Let $\mathcal{G} = (G, *_i, i \in I)$ be a semigroup system. Define a product of semigroup operations,

$$g(*_i *_j)h = (g *_i h) *_j h, \quad g, h \in G.$$

Find necessary and sufficient conditions under which $(G, *_i *_j)$ is a semigroup.

GROUP SYSTEMS AND MULTI-GROUPS

Let $\mathcal{G} = (G, *_i, i \in I)$ is semigroup (monoid, group) system, where $(G, *_i)$ is a semigroup (monoid, group) for any $i \in I$.

Definition 2. We will call $\mathcal{G}$ by multi-semigroup (multi-monoid, multi-group) if the operations are connected by the following condition

$$(a *_i b) *_j c = a *_i (b *_j c), \quad a, b, c \in G, \quad i, j \in I.$$

GROUP SYSTEMS AND MULTI-GROUPS

Let us fix a matrix $M \in M_n(k)$ and define a set of multiplications $*_{t_0,t_1} : M_n(k) \times M_n(k) \to M_n(k)$,

$$A *_{t_0,t_1} B = t_0 AB + t_1 AMB, \quad t_0, t_1 \in k.$$

V.G. Bardakov, T.A. Kozlovskaya, D.V. Talalaev, *Groups and quandles: n-valued and n-tuple*, in progress.

PROPOSITION 2.
1) $(M_n(k), *_{1,t_1}, t_1 \in k)$ is a multi-semigroup.
2) If $detM \neq 0$ then $(GL_n(k), *_{1,0}, *_{0,1})$ is a multi-group.

GROUP SYSTEMS AND MULTI-GROUPS

Let $M_n(\Bbbk)$ be a set of $n \times n$ matrices over a field $\Bbbk$.

$$A *_{s,t,M_1,M_2} B = sAM_1B + tAM_2B, \quad s, t \in \Bbbk, \quad M_1, M_2 \in M_n(\Bbbk).$$

V.G. Bardakov, T. A. Kozlovskaya, D.V. Talalaev, *Groups and quandles: n-valued and n-tuple*, in progress.

QUESTION 1. What can we say on this product? What algebraic systems one can construct using these operations? Is there connection of these operations with non-standard matrix multiplications which were studied in

V. G. Bardakov, A. A. Simonov, *Rings and groups of matrices with a nonstandard product*, (Russian) Sibirsk. Mat. Zh. 53 (2013), no. 4, 741–750; translation in Siberian Math. J. 54 (2013), no. 3, 393–405.

GROUP SYSTEMS AND MULTI-GROUPS

THEOREM 1. The groupoid system $(M_n(\mathbb{k}), *_{s,t,M_1,M_2}, s, t \in \mathbb{k}, M_1, M_2 \in M_n(\mathbb{k}))$ is a multi-semigroup.

T.A. Kozlovskaya, Multi-groups, to appear.

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QUESTION 2. Under which conditions this multi-semigroup is multi-group?

GROUP SYSTEMS AND MULTI-GROUPS

THEOREM 2.

1) Let $M \in M_n(\mathbb{k})$, $detM \neq 0$. Then $(GL_n(\mathbb{k}), *_M)$ is a group with the product $A *_M B = AMB$, with the unit element $E^{(*_M)} = M^{-1}$ and the inverse $\bar{A}^{(*_M)} = M^{-1}A^{-1}M^{-1}$.

2) The algebraic system $(GL_n(\mathbb{k}), *_M, M \in GL_n(\mathbb{k}))$ is a multi-group.

T.A. Kozlovskaya, Multi-groups, to appear.

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RACK SYSTEMS

Definition 3. A quandle is a non-empty set Q with a binary operation $(x, y) \mapsto x * y$ satisfying the following axioms:

(Q1) $x * x = x$ for all $x \in Q$,

(Q2) for any $x, y \in Q$ there exists a unique $z \in Q$ such that $x = z * y$,

(Q3) $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in Q$.

An algebraic system satisfying only (Q2) and (Q3) is called a rack.

S. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.), 119 (161), no. 1 (9) (1982), 78–88 (in Russian).

D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra, 23, no. 1 (1982), 37–65.

QUANDLES

EXAMPLE 1. If G is a group, m is an integer, then the binary operation $a *_m b = b^{-m}\, a\, b^m$ turns G into the quandle $\mathrm{Conj}_m(G)$ called the m-conjugation quandle on G.

If $m = 1$, this quandle is called conjugation quandle.

EXAMPLE 2. A group G with the binary operation $a * b = ba^{-1}b$ turns the set G into the quandle $\mathrm{Core}(G)$ called the core quandle of G.

If $G = \mathbb{Z}_n$ is the cyclic group of order n, then $\mathrm{Core}(\mathbb{Z})$ is called the dihedral quandle and denoted by R_n.

EXAMPLE 3. Let G be a group and $\varphi \in \mathrm{Aut}(G)$. Then the set G with binary operation $a *_\varphi b = \varphi(ab^{-1})b$ forms a quandle $\mathrm{Alex}(G,\ \varphi)$ referred as the generalized Alexander quandle of G with respect to φ.

RACK SYSTEMS

Let $Q = \mathrm{GL}_n(\mathbb{k})$, $\varphi \in \mathrm{Aut}(\mathrm{GL}_n(\mathbb{k}))$. Define a quandle system $(Q, *_\varphi, \varphi \in \mathrm{Aut}(GL_n(\mathbb{k})))$

QUESTION 3. What rack (quandle) systems one can defines on $V \times G$, where V is a vector space of dimension n over a field $\mathbb{k}$, G is a subgroup of $\mathrm{GL}_n(\mathbb{k})$.

QUANDLE SYSTEMS

V. G. Bardakov, D. A. Fedoseev, *Multiplication of quandle structures*, arXiv:2204.12571.

They considered quandle system $\mathcal{Q} = (Q, *_i, i \in I)$, where $(Q, *_i)$ is a quandle for any $i \in I$ and defined a multiplication $*_i *_j$ by the rule

$$p(*_i *_j)q = (p *_i q) *_j q, \quad p, q \in Q.$$

In the general case the algebraic system $(Q, *_i *_j)$ is not a quandle, but if the operations are satisfied the axioms

$$(x *_i y) *_j z = (x *_j z) *_i (y *_j z), \ \ (x *_j y) *_i z = (x *_i z) *_j (y *_i z), \quad x, y, z \in Q,$$

then $(Q, *_i *_j)$ and $(Q, *_j *_i)$ are quandles.

QUANDLE SYSTEMS

V. Turaev, Multi-quandles of topological pairs, arXiv:2205.00951

Definition 4. Quandle systems which satisfy axioms:

$$(x *_i y) *_j z = (x *_j z) *_i (y *_j z),$$

$$(x *_j y) *_i z = (x *_i z) *_j (y *_i z), \quad x, y, z \in Q,$$

for all $i, j \in I$ are called multi-quandles.

V. Turaev gave a topological interpretation of multi-quandles.

RACK SYSTEMS

THEOREM 3. Let $Q = (V, G)$, where V is a vector space of dimension n over a field $\mathbb{k}$, G is a subgroup of $\mathrm{GL}_n(\mathbb{k})$. Then the algebraic system $(Q, \circ_n, n \in \mathbb{Z})$, where

$$(a, A) \circ_n (b, B) = (A^n b, A^n B A^{-n}), \quad a, b \in V, \quad A, B \in G$$

satisfies the axioms:

1) Left self-distributivity,

$$(a, A) \circ_n ((b, B) \circ_n (c, C)) = ((a, A) \circ_n (b, B)) \circ_n ((a, A) \circ_n (c, C)), \;\; a, b, c \in V, \;\; A$$

2) Left divisibility,

for any (a, A), $(b, B) \in Q$ there is unique $(u, X) \in Q$ such that

$$(a, A) \circ_n (u, X) = (b, B).$$

T.A. Kozlovskaya, Multi-groups, to appear.

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RACK SYSTEMS

COROLLARY 1. The algebraic system $(Q, \circ_n^{op}, n \in \mathbb{Z})$, where the opposite operations are defined by the rules

$$(a, A) \circ_n^{op} (b, B) = (b, B) \circ_n (a, A)$$

is a rack system.

T.A. Kozlovskaya, *Multi-groups*, to appear.

CONTENTS

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V.G. Bardakov, T.A. Kozlovskaya, Multivalued groups and Newton polyhedron. htttps://arxiv.org/abs/2305.07261.

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n-VALUED GROUPS

V. M. Buchstaber, S. P. Novikov, Formal groups, power systems and Adams operators, Russian Mat. Sb. (N.S.), 84 (126), 1971, 81–118.

An n-valued multiplication on X is a map

$$\mu : X \times X \to (X)^n = Sym^n X,$$

$$\mu(x, y) = x * y = [z_1, z_2, \dots, z_n], \quad z_k = (x * y)_k,$$

where $(X)^n = Sym^n X$ is the n-th symmetric power of X.

n-VALUED GROUPS

Definition 1. The map μ defines n-valued group $\mathcal{X} = (X, \mu, e, inv)$ on X if the next axioms hold:
Associativity. The n^2-sets:

$$[x*(y*z)_1, x*(y*z)_2, \dots, x*(y*z)_n], \ [(x*y)_1*z, (x*y)_2*z, \dots, (x*y)_n*z]$$

coincide for all $x, y, z \in X$.
Unit. There exists an element $e \in X$ such that

$$e * x = x * e = [x, x, \dots, x]$$

for all $x \in X$.
Inverse. It is defined a map $inv : X \to X$ such that

$$e \in inv(x) * x \qquad \text{and} \qquad e \in x * inv(x)$$

for all $x \in X$.

2-VALUED GROUP STRUCTURE ON $\mathbb{Z}_+$.

EXAMPLE 1. (Buchstaber-Novikov group)

$$\mu : \mathbb{Z}_+ \times \mathbb{Z}_+ \to (\mathbb{Z}_+)^2 : \quad x * y = [x + y, |x - y|], \quad x, y \in \mathbb{Z}_+$$

Unit:

$$e = 0.$$

Inverse:

$$inv(x) = x.$$

Associativity:

$$[x + y + z, |x - y - z|, x + |y - z|, |x - |y - z||]$$

and

$$[x + y + z, |x + y - z|, |x - y| + z, ||x - y| - z|]$$

coincide for all $x, y, z \in \mathbb{Z}_+$.

COSET GROUPS

THEOREM 1.

Let G be a group with multiplication m, $A \leq \mathrm{Aut}(G)$ and $|A| = n$. The set of orbits $X = G/A$ can be equipped with an n-multiplication

$$\mu\colon X \times X \to (X)^n, \ \ \mu(x, y) = \pi(m(\pi^{-1}(x), \pi^{-1}(y))),$$

where $\pi\colon G \to X$ is the canonical projection. Then X with multiplication μ is a n-valued group with the unit $e_X = \pi(e_G)$ and the inverse $inv(x)$ of $x \in X$ is $\pi((\pi^{-1}(x))^{-1})$. This group is called the coset group of (G, A).

V. M. Buchstaber, n-valued groups: theory and applications, Moscow Math. J., 6, no. 1 (2006), 57–84.

COSET GROUPS

EXAMPLE $1'$. (Buchstaber-Novikov group)
Let $G = \mathbb{Z}$ be the infinite cyclic group, $A = \{\mathrm{id}, -\mathrm{id}\}$ a subgroup of its automorphisms of order 2. The set of orbits

$$X = \{\{0\}, \{\pm a\} \mid a \in \mathbb{N}\}$$

can be identified with the set of non-negative integers $\mathbb{Z}_+$ with the 2–valued multiplication

$$\mu\colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to (\mathbb{Z}_+)^2, \quad x * y = [x + y, |x - y|], \quad x, y \in \mathbb{Z}_+.$$

Hence, Buchstaber-Novikov group is a coset group.

DOUBLE COSET GROUPS

THEOREM 2. Let G is a group and H - its subgroup of cardinality n. Denote by X the space of double coset classes $H\backslash G/H$. Define the n-valued multiplication $\mu\colon X \times X \to (X)^n$ by the formula

$$\mu(x, y) = \{Hg_1H\} * \{Hg_2H\} = [\{Hg_1hg_2H\}, \quad h \in H].$$

and the inverse element $inv_X(x) = \{Hg^{-1}H\}$, where $x = \{HgH\}$. This construction for n-valued groups called a double coset group.

V. M. Buchstaber, n-valued groups: theory and applications, Moscow Math. J., 6, no. 1 (2006), 57–84.

DOUBLE COSET GROUPS

EXAMPLE 2. Let $S_3 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = 1 \rangle$ be the symmetric group and $S_2 = \langle s_1 \rangle$ be its subgroup of order 2. Then

$$S_2 \backslash S_3 / S_2 = \{\{1, s_1\}, \{s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}\}.$$

If we denote the classes $[1] = \{1, s_1\}$ and $[s_2] = \{s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$, then the 2-valued product is given by

$$[1] * [1] = \{[1], [1]\}, \quad [1] * [s_2] = [s_2] * [1] = \{[s_2], [1]\}, \quad [s_2] * [s_2] = \{[1], [s_2]\}.$$

Hence, $inv[s_2] = [s_2]$.

n-VALUED GROUP STRUCTURE ON $\mathbb{C}$

EXAMPLE 3. Let μ be the multiplication

$$\mu\colon \mathbb{C}\times\mathbb{C}\to(\mathbb{C})^n: \quad x*y=[(\sqrt[n]{x}+\epsilon^r\sqrt[n]{y})^n, \quad 1\le r\le n],$$

where ϵ is a primitive n-th root of unity.
Unit: $e=0$.
Inverse: $inv(x)=(-1)^n x$

n-VALUED GROUP STRUCTURE ON $\mathbb{C}$

The n-valued multiplication is described by the polynomials

$$p_n = p_n(z; x, y) = \prod_{k=1}^{n} (z - (inv(x) * inv(y))_k),$$

whence the product $x * y$ is defined by z-roots of the equation $p_n = 0$.

$$p_1 = x + y + z, \quad p_2 = (x + y + z)^2 - 4(xy + yz + zx).$$

n-VALUED GROUP STRUCTURE ON $\mathbb{C}$

The polynomials $p_n(z; x, y)$ are x, y, z–symmetric polynomials.
By the main theorem on symmetric polynomials we have

$p_1 = e_1,$
$p_2 = e_1^2 - 2^2\, e_2,$
$p_3 = e_1^3 - 3^3\, e_3,$
$p_4 = e_1^4 - 2^3\, e_1^2 e_2 + 2^4\, e_2^2 - 2^7\, e_1 e_3,$
$p_5 = e_1^5 - 5^4\, e_1^2 e_3 + 5^5\, e_2 e_3,$
$\dots$

where $e_1 = x + y + z, \quad e_2 = xy + yz + zx, \quad e_3 = xyz$ are elementary symmetric polynomials.

Multivalued groups and Buchstaber's questions

QUESTIONS.

(1) What is the relationship between prime factors of n and prime factors of the coefficients of the polynomials p_n?
(2) How to distinguish the monomials that have zero coefficient?
(3) How to describe the Newton polyhedron of p_n?

V. M. Buchstaber, n-valued groups: theory and applications, Moscow Math. J., 6, no. 1 (2006), 57–84.

Coefficients of p_n

We can present p_n as a polynomial on the elementary symmetric polynomials e_1, e_2, and e_3,

$$p_n = \sum_{\substack{k_1 \geq k_2 \geq k_3 \geq 0 \\ k_1+k_2+k_3=n}} A_{k_1,k_2,k_3} e_1^{k_1-k_2} e_2^{k_2-k_3} e_3^{k_3} \in \mathbb{Z}[e_1, e_2, e_3].$$

The main problem is to find the coefficients A_{k_1,k_2,k_3}.
We can write p_n in the form

$$p_n = \prod_{k=1}^{n} (z - ((inv(x) * inv(y))_k) = \prod_{k=1}^{n} (z - ((-1)^n x * (-1)^n y)_k) =$$

$$= \prod_{k=1}^{n} \left(z - \left(\sqrt[n]{(-1)^n x} + \epsilon^k \sqrt[n]{(-1)^n y}\right)^n\right).$$

Coefficients of p_n

If $y = 0$, then

$$\bar{p}_n = p_n(z; x, 0) = \prod_{k=1}^{n} \left(z - (\sqrt[n]{(-1)^n x})^n \right) =$$

$$= \prod_{k=1}^{n} (z - (-1)^n x) = (z - (-1)^n x)^n \,.$$

Denote by

$$\bar{e}_1 = e_1(z; x, 0) = x + z, \quad \bar{e}_2 = e_2(z; x, 0) = zx.$$

We see that $e_3(z; x, 0) = 0$.

Coefficients of p_n

THEOREM 1.

1) If n is odd, then all $A_{k_1,k_2,0}$, $k_2 \neq 0$, are zero, i.e. in this case p_n does not contains monomials $e_1^i e_2^j$, $j > 0$.
2) If $n = 2k$ is even, then the coefficient $A_{2k-i,i,0}$ at $e_1^{2(k-i)} e_2^i$, is equal to

$$A_{2k-i,i,0} = (-4)^i C_k^i = (-4)^i \frac{k!}{i!(k-i)!}, \quad i = 1, 2, \ldots, k.$$

This gives particular answer on the first two questions.

V.G. Bardakov, T.A. Kozlovskaya, *Multivalued groups and Newton polyhedron*, https://arxiv.org/pdf/2305.07261.pdf

COEFFICIENTS OF p_n

EXAMPLE.

From this theorem follows that for even n hold

$\bar{p}_2 = \bar{e}_1^2 - 2^2\,\bar{e}_2,$

$\bar{p}_4 = \bar{e}_1^4 - 2^3\,\bar{e}_1^2\bar{e}_2 + 2^4\,\bar{e}_2^2,$

$\bar{p}_6 = \bar{e}_1^6 - 2^2\cdot 3\,\bar{e}_1^4\bar{e}_2 + 2^4\cdot 3\,\bar{e}_1^2\bar{e}_2^2 - 2^6\,\bar{e}_2^3,$

$\bar{p}_8 = \bar{e}_1^8 - 2^4\,\bar{e}_1^6\bar{e}_2 + 2^5\cdot 3\,\bar{e}_1^4\bar{e}_2^2 - 2^8\,\bar{e}_1^2\bar{e}_2^3 + 2^8\,\bar{e}_2^4.$

It is easy to see that for even n all coefficients of $\bar{p}_n$ except the coefficient at $\bar{e}_1^n$ are even. It is not true for polynomials p_n, as example p_6 shows. We can formulate

CONJECTURE.

1) If $n = p^m$ is a power of a prime p, then all coefficients, except the coefficient at e_1^n are divided into p. 2) If n is even, then all coefficients A_{k_1,k_2,k_3} are non-zero.

Newton polyhedron

Definition 1. Let

$$f = f(x_1, x_2, \dots, x_n) = \sum a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n} \in \mathbb{Z}[x_1, x_2, \dots, x_n]$$

be a polynomial with integer coefficients. Denote by I_f the set of multi indexes $(i_1, \dots, i_n)$ such that $a_{i_1 \dots i_n} \neq 0$. The convex hull

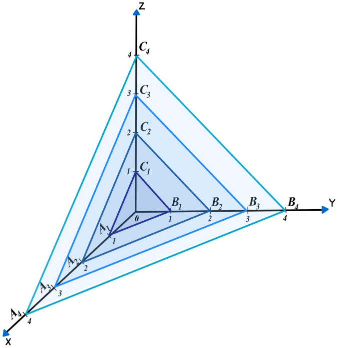
$$N_f = Conv(I_f) \subset \mathbb{R}^n$$

is said to be a Newton polyhedron of f.
To find the Newton polyhedrons for the polynomials p_n, consider them for small n,
$p_1 = x + y + z,$
$p_2 = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx,$
$p_3 = (z + x + y)^3 - 27xyz,$
$p_4 = ((x + y + z)^2 - 4(xy + yz + zx))^2 - 2^7(x + y + z)xyz = p_2^2 - 2^7 p_1 xyz.$

Newton polyhedron

Denote by $N_i \subset \mathbb{R}^3$ the Newton polyhedron for p_i.

For i=1,2,3,4, N_i is the right triangle $A_iB_iC_i$ with the vertices $A_i = (i,0,0)$, $B_i = (0,i,0)$, $C_i = (0,0,i)$.



Definition 2. Let k be a positive integer. The standard n-simplex of size k is the subset of $\mathbb{R}^{n+1}$ given by

$$k\Delta^n = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = k \text{ and } t_i \geq 0 \text{ for } i = 0, 1, \ldots, n \right\}.$$

We shall call the standard n-simplex of size k by $k\Delta^n$-simplex.

For $k = 1$ we get the definition of the standard n-simplex.

THEOREM 2.

Let $f = f(x_1, x_2, \ldots, x_n) \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$ be a symmetric homogeneous polynomial of degree k, which contains a monomial ax_1^k for some non-zero a. Then its Newton polyhedron N_f is the $k\Delta^{n-1}$-simplex.

V.G. Bardakov, T.A. Kozlovskaya, *Multivalued groups and Newton polyhedron*, https://arxiv.org/pdf/2305.07261.pdf

Newton polyhedron

The polynomial p_k is homogeneous and has the form $p_k = e_1^k + \dots$.
Hence, from the theorem follows an answer on the third question of V. M. Buchstaber.

COROLLARY.
The Newton polyhedron for $p_k(z, x, y)$, $k \geq 1$, *is the* $k\Delta^2$*-simplex that is a right triangle with sides of length* $\sqrt{2}\,k$.

V.G. Bardakov, T.A. Kozlovskaya, *Multivalued groups and Newton polyhedron*, https://arxiv.org/pdf/2305.07261.pdf

Some open questions

1. Let $f = f(x_1, x_2, \dots, x_n) \in \mathbb{Z}[x_1, x_2, \dots, x_n]$ be a symmetric polynomial, N_f is its Newton polyhedron. Let us present f as a polynomial on elementary symmetric polynomial, $f = F[e_1, e_2, \dots, e_n] \in \mathbb{Z}[e_1, e_2, \dots, e_n]$ and construct its Newton polyhedron N_F. What is connection between N_f and N_F?
2. Let $f = f(x_1, x_2, \dots, x_n) \in \mathbb{Z}[x_1, x_2, \dots, x_n]$ be a symmetric polynomial of degree k, which does not contain x_1^k. What can we say on its Newton polyhedron N_f?
3. Construct a theory of extensions multivalued groups.
4. Is it possible to define (co)homology for multivalued groups?

CONTENTS

V.G. Bardakov, T.A. Kozlovskaya, D.V. Talalaev, Groups and quandles: n-valued and n-tuple, in progress.

Let X be a non-empty set with n-valued multiplications $*$ and $\bar{*}$:

$$x * y = [(x * y)_1, (x * y)_2, \ldots, (x * y)_n],$$
$$x \bar{*} y = [(x \bar{*} y)_1, (x \bar{*} y)_2, \ldots, (x \bar{*} y)_n], \quad x, y \in X,$$

i.e. $x * y$ and $x \bar{*} y$ are multisets in $Sym^n(X)$.

Definition 1. n-Valued rack is a triple $\mathcal{X} = (X, *, \bar{*})$ such that the following axioms hold:

(M1) Invertibility: for any $x, y \in X$ the element x lies in the n^2-multiset $(x * y)\bar{*}y$ and in the n^2-multiset $(x\bar{*}y) * y$;
(M2) Self-distributivity: for any $x, y, z \in X$ the n^2-multiset $(x * y) * z$ is a subset of the n^3-multiset $(x * z) * (y * z)$.

An (n-valued) rack $\mathcal{X} = (X, *, \bar{*})$ is said to be an n-valued quandle if the next axiom holds
(M3) Idempotency: for any $x \in X$ the multiset $x * x$ contains x.

EXAMPLE.

Let G be a group and I is a subset of integer numbers. For any $i \in I$ define i-conjugation quandle $Conj_i(G)$ on G. If I is finite and contains n elements, we can define n-valued multiplication

$$g * h = [g *_i h, \quad i \in I], \quad g, h \in G.$$

In this case $g \bar{*} h = [g \bar{*}_i h, \quad i \in I]$ and it is easy to check that it is an n-valued quandle.

This algebraic system $(G, \{*_i\}_{i \in I})$ is called n-multi-quandle.

V. Turaev, Multi-quandles of topological pairs, arXiv:2205.00951

Coset construction

Let Q be a quandle with multiplication m and its inverse $\bar{m}$ that means that for any $q, h \in Q$ the following holds

$$\bar{m}(m(q,h),h) = m(\bar{m}(q,h),h) = q.$$

Further, let $A \subset \mathrm{Aut}(Q)$ be a subgroup of order n. Then the set of orbits $X = Q/A$ can be equipped with an n-multiplication

$$\mu \colon X \times X \to (X)^n$$

by the rule

$$\mu(x,y) = x * y = \pi(m(\pi^{-1}(x), \pi^{-1}(y))),$$

where $\pi \colon Q \to X$ is the canonical projection.

THEOREM 1.

The multiplication μ defines an n-valued quandle structure on the orbit space $X = Q/A$, called the *coset n-valued quandle* of (G, A), with the inverse operation

$$x \bar{*} y = \pi(\bar{m}(\pi^{-1}(x), \pi^{-1}(y))).$$

V.G. Bardakov, T.A. Kozlovskaya, D.V. Talalaev, *Groups and quandles: n-valued and n-tuple*, in progress.

Coset construction

EXAMPLE 1.

Let us consider the conjugation quandle $Q = Conj(S_3)$ for the group S_3 and subgroup of inner automorphisms $A \subset Inn(Q)$ that is generated by multiplication on s_1. It is evident that $A \cong S_2$. The orbits of Q under the action of A are

$$1, s_1, \{s_2, s_1 s_2 s_1\}, \{s_1 s_2, s_2 s_1\}.$$

Let us denote these classes by $(x_0, \dots, x_3)$. The multiplication table for this 2-quandle is given by

Q/A	x_0	x_1	x_2	x_3
x_0	$[x_0, x_0]$	$[x_0, x_0]$	$[x_0, x_0]$	$[x_0, x_0]$
x_1	$[x_1, x_1]$	$[x_1, x_1]$	$[x_2, x_2]$	$[x_2, x_2]$
x_2	$[x_2, x_2]$	$[x_2, x_2]$	$[x_1, x_2]$	$[x_1, x_2]$
x_3	$[x_3, x_3]$	$[x_3, x_3]$	$[x_3, x_3]$	$[x_3, x_3]$

.

EXAMPLE 2. Let us consider the finite cyclic group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ of order 4 and the automorphism $\varphi\colon \mathbb{Z}_4 \to \mathbb{Z}_4$,

$$\varphi(0) = 0, \quad \varphi(1) = 3, \quad \varphi(2) = 2, \quad \varphi(3) = 1.$$

This automorphism has order 2 and the 2-valued coset group $X = (\mathbb{Z}_4, \langle\varphi\rangle)$ has three elements,

$$a = \{0\}, \quad b = \{1, 3\}, \quad c = \{2\},$$

with the multiplication table

*	a	b	c
a	[a, a]	[b, b]	[c, c]
b	[b, b]	[a, c]	[b, b]
c	[c, c]	[b, b]	[a, a]

.

Hence, a is the unit element and $inv = id$ is the identity map.

Coset construction

EXAMPLE 3. Let us consider the dihedral quandle $R_4 = \{a_0, a_1, a_2, a_3\}$. This quandle has inner automorphisms $S_i = S_{a_i}$, $i = 0, 1, 2, 3$, which act by the rules

$$a_0 S_0 = a_0, \quad a_1 S_0 = a_3, \quad a_2 S_0 = a_2, \quad a_3 S_0 = a_1,$$

$$a_0 S_1 = a_2, \quad a_1 S_1 = a_1, \quad a_2 S_1 = a_0, \quad a_3 S_1 = a_3.$$

Further, $S_2 = S_0$, $S_3 = S_1$, $S_0 S_1 = S_1 S_0$ and $\mathrm{Inn}(R_4) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Consider the coset 2-valued quandle $Q = (R_4, \langle S_0 \rangle)$. This quandle has elements $A = \{a_0\}$, $B = \{a_1, a_3\}$, $C = \{a_2\}$, and the multiplication table

$*$	A	B	C
A	[A, A]	[C, C]	[A, A]
B	[B, B]	[B, B]	[B, B]
C	[C, C]	[A, A]	[C, C]

.

Ordered n-valued groups

Definition 2. An ordered n-valued group is a non-empty set G with n-tuple multiplication

$$x * y = (x *_1 y, x *_2 y, \ldots, x *_n y),$$

such that the following axioms hold:

1) associativity:

$$(x *_i y) *_j z = x *_i (y *_j z), \quad i, j \in \{1, 2, \ldots, n\},$$

for all $x, y, z \in X$.

2) unit element:

for any operation $*_i$ there exists an element $e_i \in G$ such that

$$x *_i e_i = e_i *_i x = x.$$

If $e_1 = e_2 = \ldots = e_n = e$, then we say that an ordered n-valued group has a common unit element $e \in G$. In this case for any $x \in G$ we have

$$x * e = (x *_1 e, x *_2 e, \ldots, x *_n e) = (x, x, \ldots, x),$$

$$e * x = (e *_1 x, e *_2 x, \ldots, e *_n x) = (x, x, \ldots, x);$$

3) invertibility: for any operation $*_i$ for any $x \in G$ the exists an inverse element $\bar{x}^{(i)}$ such that

$$x *_i \bar{x}^{(i)} = \bar{x}^{(i)} *_i x = e_i.$$

It is easy to see that $(G, *_i)$ is a group for any $i \in \{1, 2, \ldots, n\}$ and $(G, *_1, ..., *_n)$ is n-multi-group.

PROPOSITION 1.
Let $(G, *_1, ..., *_n)$ be an n-multi-group. If $e_i = e_j$ for some $i, j \in \{1, 2, \ldots, n\}$, then $*_i = *_j$.

Ordered n-valued quandles

THEOREM 2. Let X be a non empty set with n quandle operations $*_i$ such that for any pair $1 \leq i, j \leq n$ the following relations hold

$$(x *_i y) *_j z = (x *_j z) *_i (y *_j z), \quad x, y, z \in X.$$

Then an n-valued multiplication

$$x * y = [x *_1 y, x *_2 y, \ldots, x *_n y]$$

defines an n-valued quandle structure on X.

V.G. Bardakov, T.A. Kozlovskaya, D.V. Talalaev, *Groups and quandles: n-valued and n-tuple*, in progress.

In particular, if we take the powers on quandle operations, then we get

COROLLARY. Let $(Q, \circ)$ be an n-quandle (it means that for any $q, h \in Q$ we have $q *^n h = q$). Let us define an n-valued multiplication on Q by the formula

$$x * y = [x \circ y, x \circ^2 y, \ldots, x \circ^n y], \quad x, y \in Q.$$

Then $(Q, *)$ is an n-valued quandle.

V.G. Bardakov, T.A. Kozlovskaya, D.V. Talalaev, Groups and quandles: n-valued and n-tuple, in progress.

THANK
YOU