О представлении группы виртуальных кос, связанном с кластерными алгебрами

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K. Hikami, R. Inoue, Braids, complex volume and cluster algebra, Algebraic and Geometric Topology 15 (2015), 2175–2194.

A cluster seed $(\boldsymbol{x}, \mathbf{B})$ is a pair of

- a cluster variable $\boldsymbol{x} = (x_1, \dots, x_N)$: an N-tuple of algebraically independent variables,
- an exchange matrix $\mathbf{B} = (b_{ij})$: an $N \times N$ skew symmetric integer matrix.

For each $k = 1, \ldots, N$, we define the mutation μ_k of $(\boldsymbol{x}, \mathbf{B})$ by

$$\mu_k\,(\boldsymbol{x}, \mathbf{B}) = (\widetilde{\boldsymbol{x}}, \widetilde{\mathbf{B}}),$$

where

- $\widetilde{\boldsymbol{x}} = (\widetilde{x}_1, \ldots, \widetilde{x}_N)$ is

$$\widetilde{x}_i = \begin{cases} x_i, & \text{for } i \neq k, \\[2ex] \dfrac{1}{x_k} \left(\displaystyle\prod_{j : b_{jk} > 0} {x_j}^{b_{jk}} + \prod_{j : b_{jk} < 0} {x_j}^{-b_{jk}} \right), & \text{for } i = k, \end{cases}$$

- $\widetilde{\mathbf{B}} = (\widetilde{b}_{ij})$ is

$$\widetilde{b}_{ij} = \begin{cases} -b_{ij}, & \text{for } i = k \text{ or } j = k, \\[2ex] b_{ij} + \dfrac{\left|b_{ik}\right| b_{kj} + b_{ik} \left|b_{kj}\right|}{2}, & \text{otherwise.} \end{cases}$$

In terms of the cluster variable $\boldsymbol{x}$, we introduce the y-variable, $\boldsymbol{y} = (y_1, \ldots, y_N)$, defined by

$$y_j = \prod_k {x_k}^{b_{kj}}.$$

The mutation μ_k induces a mutation of a pair $(\boldsymbol{y}, \mathbf{B})$:

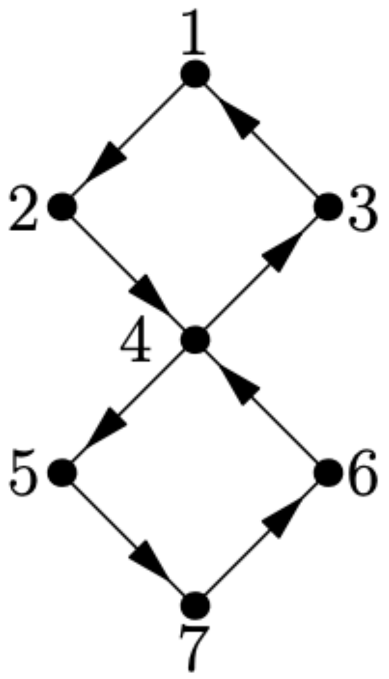
$$\mu_k(\boldsymbol{y}, \mathbf{B}) = (\widetilde{\boldsymbol{y}}, \widetilde{\mathbf{B}}),$$

where $\widetilde{\mathbf{B}}$ is (2.3), and $\widetilde{\boldsymbol{y}} = (\widetilde{y}_1, \ldots, \widetilde{y}_N)$ with $\widetilde{y}_j = \prod_k \widetilde{x}_k{}^{\widetilde{b}_{kj}}$ is given by

$$\widetilde{y}_i = \begin{cases} y_k{}^{-1}, & \text{for } i = k, \\ y_i \, \left(1 + y_k{}^{-1}\right)^{-b_{ki}}, & \text{for } i \neq k,\ b_{ki} \geq 0, \\ y_i \left(1 + y_k\right)^{-b_{ki}}, & \text{for } i \neq k,\ b_{ki} \leq 0. \end{cases}$$

7 variables

Consider case when we have 7 variables $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ and matrix $\mathbf{B}$ is taken as an adjacency matrix of the oriented graph below.



$$\mathsf{R} = s_{3,5}\, s_{2,5}\, s_{3,6}\, \mu_4\, \mu_2\, \mu_6\, \mu_4.$$

$$\mathsf{R}^{-1} = s_{3,6}\, s_{2,5}\, s_{3,5}\, \mu_4\, \mu_5\, \mu_3\, \mu_4.$$

The permutations are included in the R-operator so that the exchange matrix $\mathbf{B}$ is invariant under R.

Action of R on x-variables

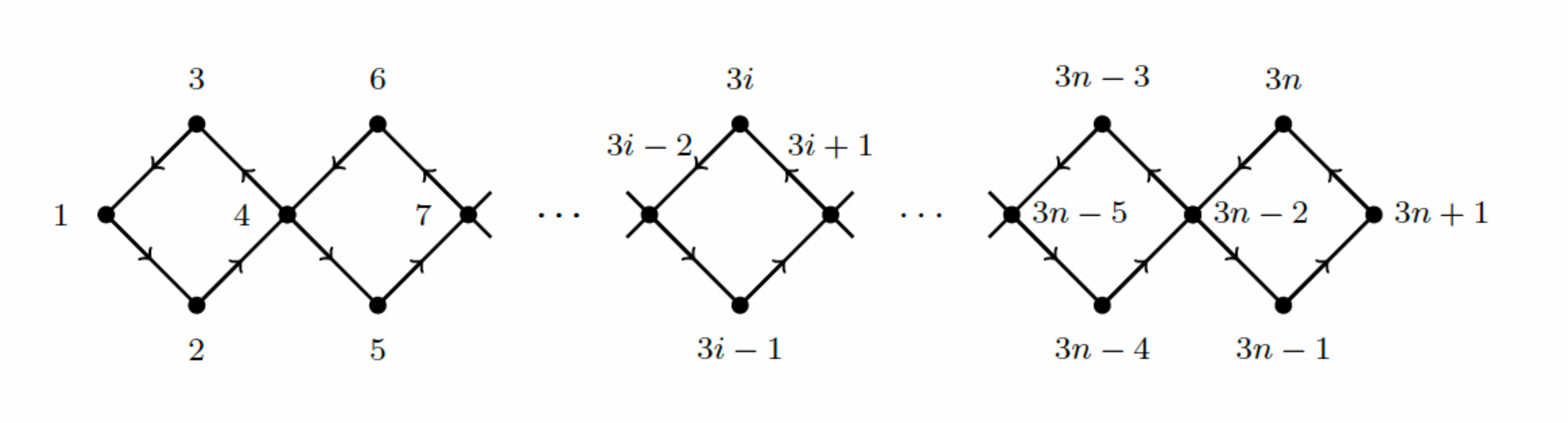
$$\mathsf{R}(\boldsymbol{x}) = \begin{pmatrix} x_1 \\ x_5 \\ \dfrac{x_1\,x_3\,x_5 + x_3\,x_4\,x_5 + x_1\,x_2\,x_6}{x_2\,x_4} \\ \dfrac{x_1\,x_3\,x_4\,x_5 + x_3\,{x_4}^2\,x_5 + x_1\,x_3\,x_5\,x_7 + x_3\,x_4\,x_5\,x_7 + x_1\,x_2\,x_6\,x_7}{x_2\,x_4\,x_6} \\ \dfrac{x_3\,x_4\,x_5 + x_3\,x_5\,x_7 + x_2\,x_6\,x_7}{x_4\,x_6} \\ x_3 \\ x_7 \end{pmatrix}^{\top},$$

$$\mathsf{R}^{-1}(\boldsymbol{x}) = \begin{pmatrix} x_1 \\ \dfrac{x_1\,x_3\,x_5 + x_1\,x_2\,x_6 + x_2\,x_4\,x_6}{x_3\,x_4} \\ x_6 \\ \dfrac{x_1\,x_2\,x_4\,x_6 + x_2\,{x_4}^2\,x_6 + x_1\,x_3\,x_5\,x_7 + x_1\,x_2\,x_6\,x_7 + x_2\,x_4\,x_6\,x_7}{x_3\,x_4\,x_5} \\ x_2 \\ \dfrac{x_2\,x_4\,x_6 + x_3\,x_5\,x_7 + x_2\,x_6\,x_7}{x_4\,x_5} \\ x_7 \end{pmatrix}^{\top}.$$

Action of R on y-variables

$$\mathsf{R}(\boldsymbol{y}) = \begin{pmatrix} y_1\,(1+y_2+y_2y_4) \\ \dfrac{y_2\,y_4\,y_5\,y_6}{1+y_2+y_6+y_2\,y_6+y_2\,y_4\,y_6} \\ \dfrac{1+y_2+y_6+y_2\,y_6+y_2\,y_4\,y_6}{y_2\,y_4} \\ \dfrac{y_4}{(1+y_2+y_2\,y_4)\,(1+y_6+y_4\,y_6)} \\ \dfrac{1+y_2+y_6+y_2\,y_6+y_2\,y_4\,y_6}{y_4\,y_6} \\ \dfrac{y_2\,y_3\,y_4\,y_6}{1+y_2+y_6+y_2\,y_6+y_2\,y_4\,y_6} \\ (1+y_6+y_4\,y_6)\;y_7 \end{pmatrix}^{\top}, \qquad \mathsf{R}^{-1}(\boldsymbol{y}) = \begin{pmatrix} \dfrac{y_1\,y_3\,y_4}{1+y_4+y_3\,y_4} \\ \dfrac{y_5}{1+y_4+y_3\,y_4+y_4\,y_5+y_3\,y_4\,y_5} \\ \dfrac{(1+y_4+y_3\,y_4+y_4\,y_5+y_3\,y_4\,y_5)\;y_6}{(1+y_4+y_3\,y_4)(1+y_4+y_4\,y_5)} \\ y_3\,y_4\,y_5 \\ y_2\,(1+y_4+y_3\,y_4+y_4\,y_5+y_3\,y_4\,y_5) \\ \dfrac{y_3}{1+y_4+y_3\,y_4+y_4\,y_5+y_3\,y_4\,y_5} \\ \dfrac{y_4\,y_5\,y_7}{1+y_4+y_4\,y_5} \end{pmatrix}^{\top}.$$

General case



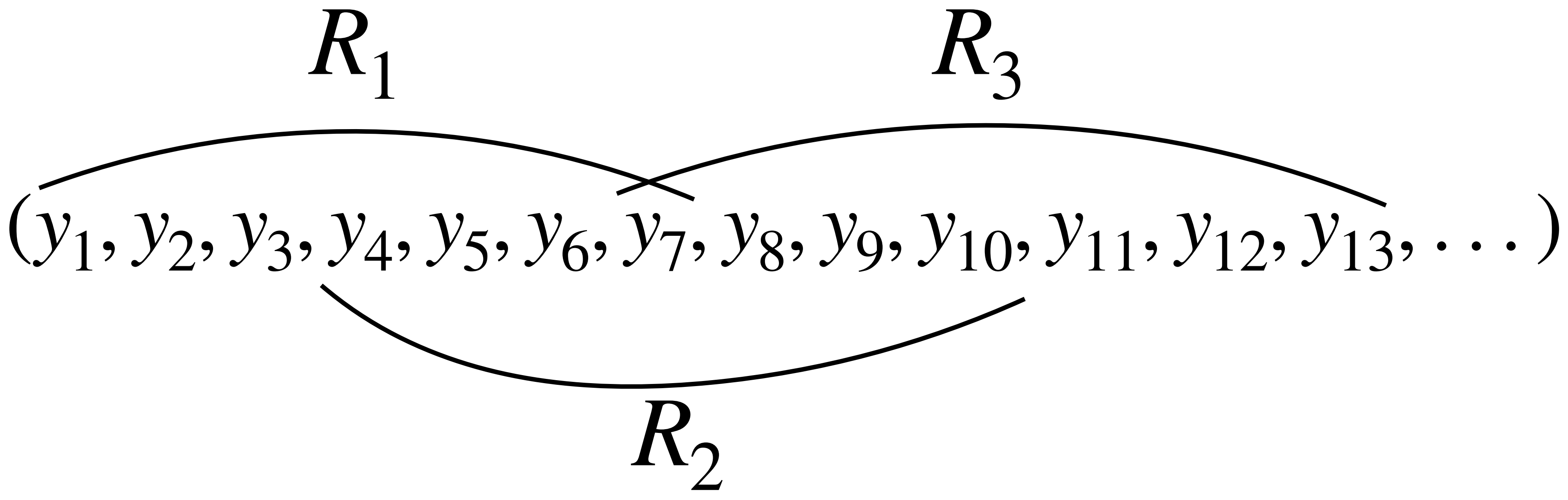
$$(x_1, x_2, x_3, x_4 \ldots, x_{3n}, x_{3n+1})$$

$$(y_1, y_2, y_3, y_4 \ldots, y_{3n}, y_{3n+1})$$

And $\mathbf{B}$ is taken as an adjacency matrix of the oriented graph above.

R_i operators

With $3n+1$ we have exactly $n-1$ R_i, $i = 1,\dots,n-1$, operators.



$$(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, \dots)$$

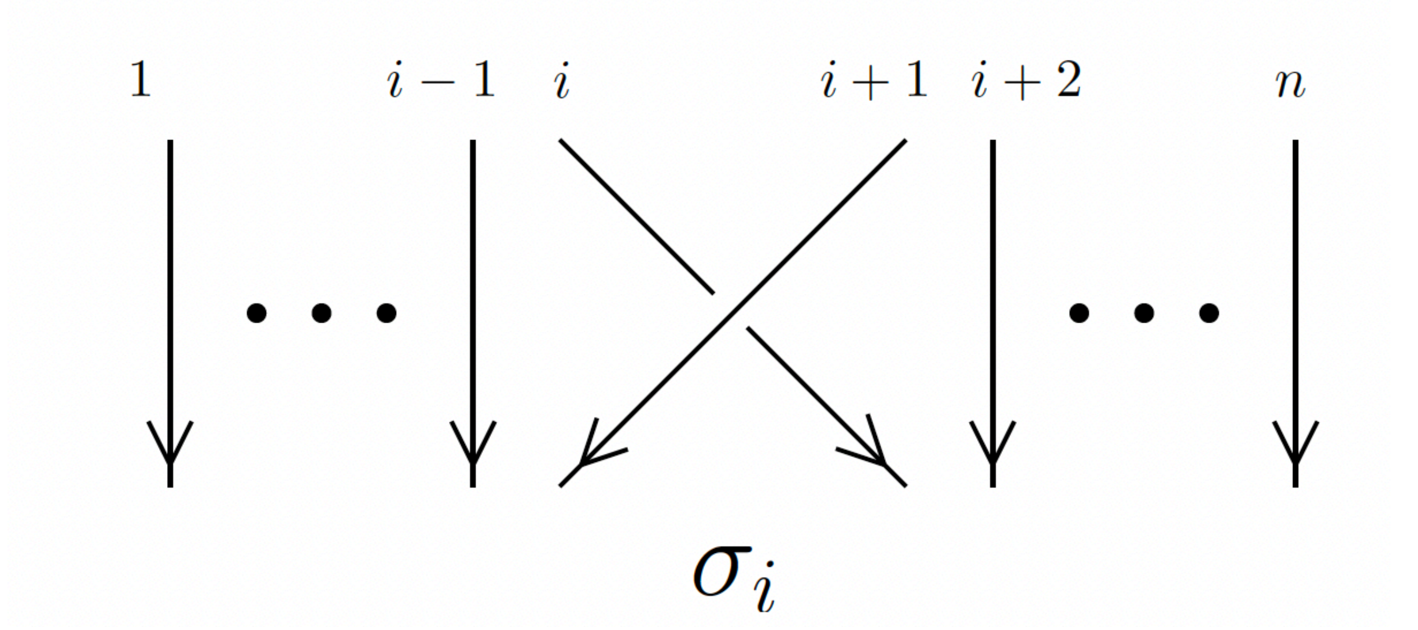
Operator R_i acts variables $y_{3i-2}, y_{3i-1}, \dots, y_{3i+4}$.

Braid group

For $n \geq 2$, the braid group B_n is defined as a group with generators $\sigma_1, \dots, \sigma_{n-1}$ and the following defining relations:

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, && |i-j| \geq 2. \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, && i = 1, 2, \dots, n-2, \end{aligned}$$

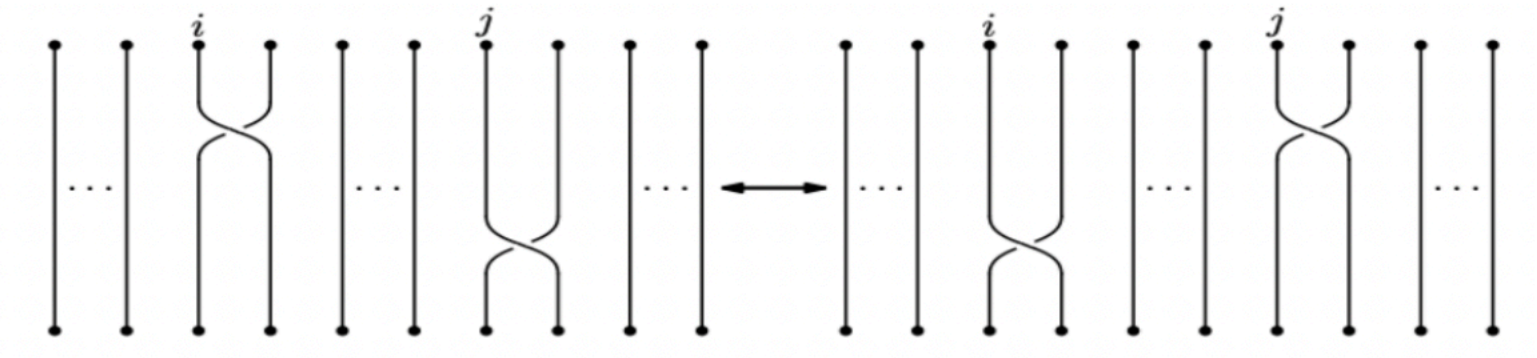
The generators σ_i have the following geometric interpretation:

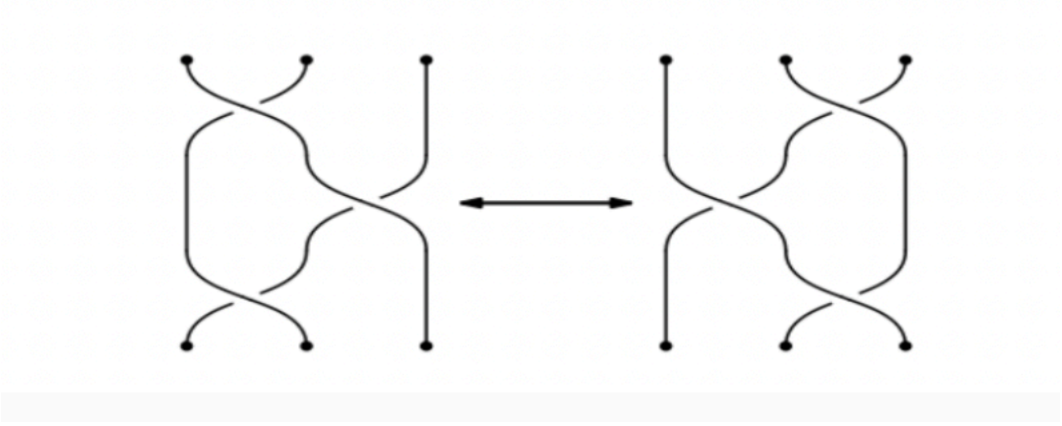


Braid group

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, & |i-j| \geq 2. \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & i = 1, 2, \ldots, n-2, \end{aligned}$$

The relations have the following geometric interpretation:





Braid group representation

Denote by $\mathbf{W_n}$ the group generated by $R_i, i = 1,..,n-1$, with composition as a group operation.
Define a map $F : B_n \to \mathbf{W_n}$ by setting

$$F(\sigma_i) = R_i.$$

Theorem (Hikami, Inoue, 2015) Map $F : B_n \to \mathbf{W_n}$ is a homomorphism.

So, the following relations are satisfied

$$\begin{aligned} R_iR_{i+1}R_i &= R_{i+1}R_iR_{+1}, && i = 1,\ldots,n-2, \\ R_iR_j &= R_jR_i, && |i-j| > 2. \end{aligned}$$

Virtual braid group

The virtual braid group VB_n on n strings is the group with two families of generators, classical and virtual, denoted by $\sigma_1, \dots, \sigma_{n-1}$ and $\rho_1, \dots, \rho_{n-1}$, with the following defining relations:

Classical relations

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, && i = 1, 2, \dots, n-2, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, && |i-j| \geq 2. \end{aligned}$$

Virtual relations

$$\begin{aligned} \rho_i \rho_{i+1} \rho_i &= \rho_{i+1} \rho_i \rho_{i+1}, && i = 1, 2, \dots, n-2, \\ \rho_i \rho_j &= \rho_j \rho_i, && |i-j| \geq 2, \\ \rho_i^2 &= 1, && i = 1, 2, \dots, n-1, \end{aligned}$$

Mixed relations

$$\begin{aligned} \sigma_i \rho_j &= \rho_j \sigma_i, && |i-j| \geq 2, \\ \rho_i \rho_{i+1} \sigma_i &= \sigma_{i+1} \rho_i \rho_{i+1}, && i = 1, 2, \dots, n-2. \end{aligned}$$

Forbidden relations

$$\begin{aligned} \rho_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\rho_{i+1}, \\ \rho_{i+1}\sigma_i\sigma_{i+1} &= \sigma_i\sigma_{i+1}\rho_i \end{aligned}$$

There is involutive non-trivial permutation T_i acting on $(y_{3i-2}, y_{3i-1}, y_{3i}, y_{3i+1}, y_{3i+2}, y_{3i+3}, y_{3i+4})$ for which the following mixed relations are satisfied

$$R_i R_{i+1} T_i = T_{i+1} R_i R_{+1}, \qquad i = 1, \ldots, n-2,$$

The idea now is not only to find suitable involution T_i, but also put some additional condition on y_i for which the relations above will be satisfied.

We also wanted the forbidden relations not to be fulfilled.

$$(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13})$$

The condition that works

$T_i(y_{3i-2}, y_{3i-1}, y_{3i}, y_{3i+1}, y_{3i+2}, y_{3i+3}, y_{3i+4}) = (y_{3i+1}, y_{3i+2}, y_{3i+3}, y_{3i-2}, y_{3i-1}, y_{3i}, y_{3i+4})$

The condition is that $y_1 = y_4 = y_7 = \ldots y_{3n+1} = -1$.

Easy to this condition is stable under the action of R_i and T_i :

Let $\mathbf{y} = (-1, y_2, y_3, -1, y_5, y_6, -1, \ldots)$, then $R_i(\mathbf{y})$ и $T_i(\mathbf{y})$ have -1 on 1st, 4th, 7th,... places.

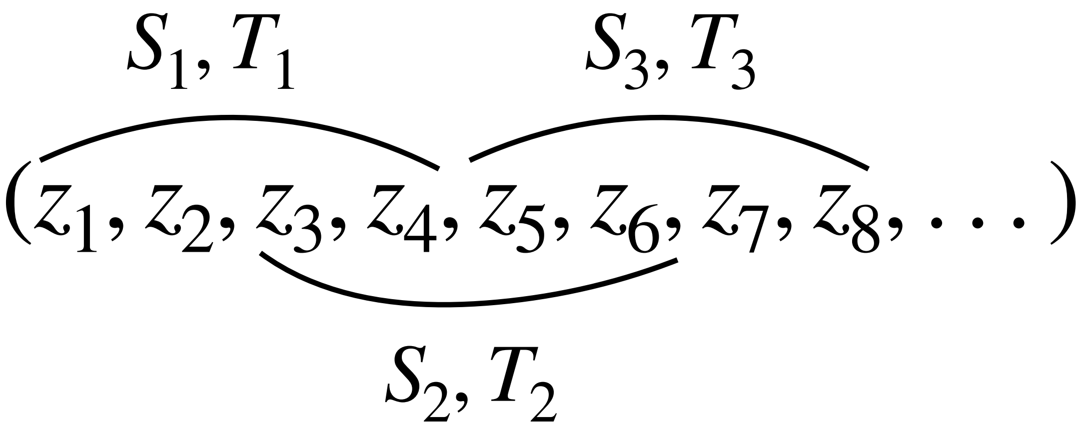
$$R_i R_{i+1} T_i = T_{i+1} R_i R_{+1}, \qquad i = 1, \ldots, n-2,$$

The final operators

At the end we have vector $\mathbf{z} = (z_1, z_2, \ldots, z_{2n})$ and two families of operators S_i and T_i, $i = 1,\ldots,n-1$.

$$S\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}^T = \begin{pmatrix} -\dfrac{z_1 z_3 z_4}{1+z_1+z_4} \\ -\dfrac{1+z_1+z_4}{z_1} \\ -\dfrac{1+z_1+z_4}{z_4} \\ -\dfrac{z_1 z_2 z_4}{1+z_1+z_4} \end{pmatrix}^T, \quad S^{-1}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}^T = \begin{pmatrix} -\dfrac{z_3}{z_2+z_3+z_2 z_3} \\ -(z_2+z_3+z_2 z_3) z_4 \\ -z_1(z_2+z_3+z_2 z_3) \\ -\dfrac{z_2}{z_2+z_3+z_2 z_3} \end{pmatrix}^T$$

$$T(z_1, z_2, z_3, z_4) = (z_3, z_4, z_1, z_2).$$



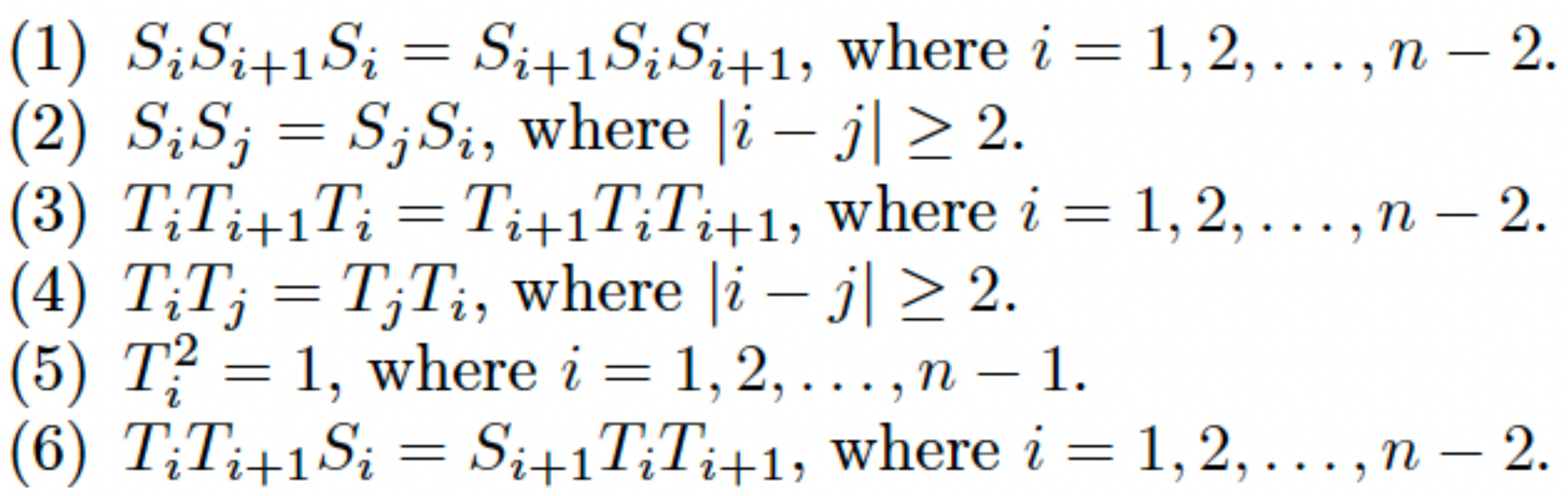
Virtual braid group representation

Denote by Θ_n the group generated by S_i и $T_i, i = 1,..,n-1$, with composition as a group operation.
Define a map $F : VB_n \to \Theta_n$ by setting

$$F(\sigma_i) = S_i, \qquad F(\rho_i) = T_i$$

Theorem Map $F : VB_n \to \Theta_n$ is a homomorphism.

So, the following relations are satisfied



(1) $S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}$, where $i = 1, 2, \ldots, n-2$.
(2) $S_i S_j = S_j S_i$, where $|i-j| \geq 2$.
(3) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, where $i = 1, 2, \ldots, n-2$.
(4) $T_i T_j = T_j T_i$, where $|i-j| \geq 2$.
(5) $T_i^2 = 1$, where $i = 1, 2, \ldots, n-1$.
(6) $T_i T_{i+1} S_i = S_{i+1} T_i T_{i+1}$, where $i = 1, 2, \ldots, n-2$.

Forbidden relations

Theorem The forbidden relations are not satisfied for S_i and T_i.

Examples

Example 1. Let $w_1 = \sigma_1\rho_1\sigma_1 \in VB_2$. The operator $F(w_1)$ $(= S_1T_1S_1)$ acts on $(1,2,2,1)$ in the following way:

$$F(w_1)(1,2,2,1) = \left(-\frac{6}{5}, -\frac{5}{3}, -\frac{5}{3}, -\frac{6}{5}\right) \neq (1,2,2,1)\,.$$

Therefore, the homomorphism F distinguishes w_1 from a trivial braid.

Example 2. It is known that a generalized Burau representation does not distinguish a braid $w_2 = (\sigma_1^2\rho_1\sigma_1^{-1}\rho_1\sigma_1^{-1}\rho_1)^2 \in VB_2$ from a trivial braid. The operator $F(w_2)$ acts on $(1,2,2,1)$ in the following way:

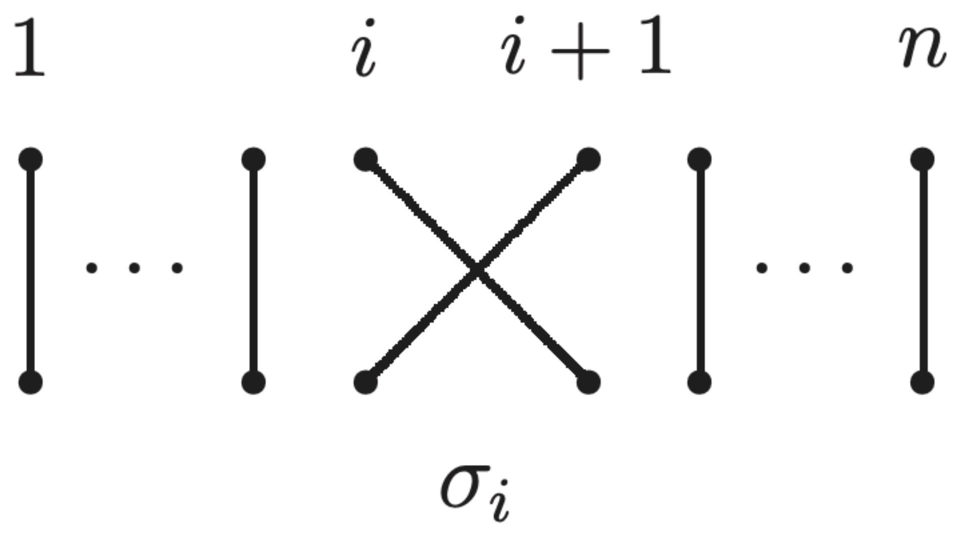
$$F(w_2)(1,2,2,1) = \left(-\frac{44}{19}, -\frac{19}{22}, -\frac{19}{22}, -\frac{44}{19}\right) \neq (1,2,2,1)\,.$$

Therefore, the homomorphism F distinguishes w_2 from a trivial braid.

Flat virtual braid group

Flat virtual braid group FVB_n is a virtual braid group VB_n with additional relations

$$\sigma_i^2 = 1, \qquad i = 1, 2, \ldots, n-1.$$



Adapting to flat braids

We noticed that if we consider vector of variables of the following form

$$\mathbf{z} = \left(z_1, \frac{1}{z_1}, z_3, \frac{1}{z_3}, \ldots, z_{2n-1}, \frac{1}{z_{2n-1}}\right)$$

Then

1) $S_i(\mathbf{z}) = (\psi_1, \dfrac{1}{\psi_1}, \psi_3, \dfrac{1}{\psi_3}, , \ldots, \psi_{2n-1}, \dfrac{1}{\psi_{2n-1}})$. And same for T_i.

2) $S_i^2(\mathbf{z}) = \mathbf{z}$

Flat virtual braid group representation

Consider $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ and define operators

$$R_i : \begin{cases} t_i \to & -\dfrac{t_i t_{i+1}}{1 + t_{i+1} + t_i t_{i+1}}, \\ t_{i+1} \to & -(1 + t_{i+1} + t_i t_{i+1}). \end{cases} \qquad V_i : \begin{cases} t_i \to & t_{i+1}, \\ t_{i+1} \to & t_i. \end{cases}$$

Denote by Ψ_n the group generated by R_i и V_i, $i = 1,..,n-1$, with composition as a group operation.
Define a map $F_{FVB} : FVB_n \to \Psi_n$ by setting

$$F(\sigma_i) = R_i, \qquad F(\rho_i) = V_i$$

Theorem Map $F_{FVB} : FVB_n \to \Psi_n$ is a homomorphism.

Example

Example 3. Let $w_3 = \sigma_2\rho_1\sigma_1\rho_2 \in FVB_3$. The operator $F_{FVB}(w_3)$ acts on $(1,2,2)$ in the following way:

$$F_{FVB}(w_3)(1,2,2) = \left(-5, \frac{4}{11}, -\frac{11}{5}\right) \neq (1,2,2)\,.$$

Therefore, the homomorphism F_{FVB} distinguishes w_3 from a trivial braid.